A SURPRISING INSTANCE OF DIVIDING

ALEX KRUCKMAN AND GABE CONANT

In any theory in which forking equals dividing for complete types (i.e., $\mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{f}} = \mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{d}}$), $a \mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{d}}_{C} b$ implies $\operatorname{acl}(Ca) \mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{d}}_{\operatorname{acl}(C)} \operatorname{acl}(Cb)$, by Proposition 2 below. In particular, this is true in every simple theory. In this note, we show that in an arbitrary theory, we cannot in general add acl to the base or the right hand side of $\mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{d}}$. That is, it is possible to have $a \mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{d}}_{C} b$ but $a \mathop{\not\smile\hskip -0.9em ^| \ }^{\mathrm{d}}_{\operatorname{acl}(C)} b$, and it follows from base monotonicity for dividing that also $a \mathop{\not\smile\hskip -0.9em ^| \ }^{\mathrm{d}}_{C} \operatorname{acl}(Cb)$. Our example is in the context of an $\aleph_0$-categorical theory T in a finite relational language. Further, we show T is SOP_3 and NSOP_4. This is the first example we are aware of in which forking is not equal to dividing for complete types in the context of an NSOP theory.

Proposition 1. *If* $a \mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{d}}_{C} b$*, then* $\operatorname{acl}(Ca) \mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{d}}_{C} b$.

Proof. Recall that $a \mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{d}}_{C} b$ if and only if for any C-indiscernible sequence I starting with b, there exists a Ca-indiscernible sequence I' with $\operatorname{tp}(I'/Cb) = \operatorname{tp}(I/Cb)$.

Assume $a \mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{d}}_{C} b$, and let I be a C-indiscernible sequence starting with b. By the characterization of dividing independence, there is a Ca-indiscernible sequence I' with $\operatorname{tp}(I'/Cb) = \operatorname{tp}(I/Cb)$. But any A-indiscernible sequence is automatically $\operatorname{acl}(A)$-indiscernible, so I' is $\operatorname{acl}(Ca)$-indiscernible. This establishes that $\operatorname{acl}(Ca) \mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{d}}_{C} b$, by the same characterization of dividing independence. $\square$

Proposition 2. *If* $a \mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{f}}_{C} b$*, then* $\operatorname{acl}(Ca) \mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{f}}_{\operatorname{acl}(C)} \operatorname{acl}(Cb)$.

Proof. Recall that $a \mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{f}}_{C} b$ if and only if for any d, there is some a' with $\operatorname{tp}(a'/Cb) = \operatorname{tp}(a/Cb)$ such that $a' \mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{d}}_{C} bd$.

Assume $a \mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{f}}_{C} b$. We first show that $\operatorname{acl}(Ca) \mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{f}}_{C} b$. Let d be an arbitrary tuple. Then there is some a' with $\operatorname{tp}(a'/Cb) = \operatorname{tp}(a/Cb)$ such that $a' \mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{d}}_{C} bd$. By Proposition 1, $\operatorname{acl}(Ca') \mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{d}}_{C} bd$. But $\operatorname{tp}(a'/Cb) = \operatorname{tp}(a/Cb)$ implies that $\operatorname{tp}(\operatorname{acl}(Ca')/Cb) = \operatorname{tp}(\operatorname{acl}(Ca)/Cb)$ (as long as we enumerate these sets by tuples in a coherent way). So $\operatorname{acl}(Ca) \mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{f}}_{C} b$.

Replacing a by an enumeration of $\operatorname{acl}(Ca)$, it suffices to show that if $a \mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{f}}_{C} b$, then $a \mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{f}}_{\operatorname{acl}(C)} \operatorname{acl}(Cb)$.

The next step is to show that $a \mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{f}}_{C} \operatorname{acl}(Cb)$. Since $a \mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{f}}_{C} b$, we can find a realization a' of a non-forking extension of $\operatorname{tp}(a/Cb)$ to a type over $\operatorname{acl}(Cb)$. Let σ be an automorphism of the monster model which fixes Cb pointwise and moves a' to a. Then σ fixes $\operatorname{acl}(Cb)$ setwise, so $a \mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{f}}_{C} \operatorname{acl}(Cb)$.

Finally, by base monotonicity, since $C \subseteq \operatorname{acl}(C) \subseteq \operatorname{acl}(Cb)$, $a \mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{f}}_{\operatorname{acl}(C)} \operatorname{acl}(Cb)$, as desired. $\square$

The example

Consider a language L' with:

- Three sorts: O, G, and C.
- Two constant symbols 0 and 1 of type C.
- A relation symbol $R\colon G \times G \times C$.
- A relation symbol $E\colon O \times G \times C$.

Let L be the sub-language of L' which omits the constant symbols 0 and 1.

Let $\mathcal{K}$ be the class of finite L'-structures satisfying the following conditions:

- $0 \neq 1$, and for all $c \in C$, $c = 0$ or $c = 1$.
- The binary relations $R(x, y, 0)$ and $R(x, y, 1)$ are disjoint graph relations on G, i.e., they are each symmetric and anti-reflexive, and for all $v, w \in G$, it is not the case that $R(v, w, 0)$ and $R(v, w, 1)$.
- For all $v, w \in G$ and $c \in C$, if $R(v, w, c)$, then there is no $o \in O$ such that $E(o, v, c)$ and $E(o, w, c)$.

It is easy to check that $\mathcal{K}$ is a Fraïssé class with free amalgamation. Let M' be the Fraïssé limit of $\mathcal{K}$, and let $T' = \mathrm{Th}(M')$. Let M and T be the reducts of L of M' and T'. We observe the following:

- T is $\aleph_0$-categorical. Indeed, it is a reduct of a Fraïssé limit of a class of finite structures in a finite language with no function symbols.
- For any $A \subseteq M$, $\mathrm{acl}(A) = A \cup C$. Indeed, $A \cup C$ is the domain of a substructure of M'. Since amalgamation in $\mathcal{K}$ is disjoint, $A \cup C$ is algebraically closed in M', so it remains algebraically closed in the reduct M.
- There is an automorphism σ of M swapping the two elements of C. Proof: back and forth.
- For any finite tuples a and b from M, $\mathrm{tp}(a) = \mathrm{tp}(b)$ if and only if there is an isomorphism of L-structures $f\colon \mathrm{acl}(a) \to \mathrm{acl}(b)$ such that $f(a) = b$. Indeed, suppose $f\colon \mathrm{acl}(a) \to \mathrm{acl}(b)$ is such an isomorphism. If f is the identity on C, then it is an isomorphism between L'-substructures of M', so it extends to an automorphism of M' moving a to b, and hence $\mathrm{tp}(a) = \mathrm{tp}(b)$. If f swaps the two elements of C, we note that $f \circ \sigma^{-1}\colon \sigma(\mathrm{acl}(a)) \to \mathrm{acl}(b)$ is an isomorphism which is the identity on C and such that $f(\sigma(a)) = b$. Extending $f \circ \sigma^{-1}$ to an automorphism of M' as above, then pre-composing with σ, we find an automorphism of M moving a to b.

By $\aleph_0$-categoricity, to understand dividing and the properties SOP_n in T, it suffices to work in M. We will call the elements of C 0 and 1, even though these constant symbols are not in L.

Claim 1. *T is* SOP_3 *and* NSOP_4.

Proof. Since T' is the theory of an $\aleph_0$-categorical Fraïssé limit with free amalgamation, it is NSOP_4. Therefore its reduct T is also NSOP_4. We will show that T is SOP_3 using the "two formula" formulation. Let x be a variable of type O, y, y' variables of type G, and z a variable of type C. Let $\varphi(x; y, y', z)$ be the formula $E(x, y, z)$, and let $\varphi'(x; y, y', z)$ be the formula $E(x, y', z)$. Let $I = (b_i, b'_i, c_i)_{i\in\omega}$ be a sequence such that $R(b'_i, b_j, 0)$ if and only if $i < j$, and $c_i = 0$ for all $i \in I$. Then for all $n \in \omega$, we have $\{\varphi(x; b_i, b'_i, c_i) \mid i < n\} \cup \{\varphi'(x; p_j, p'_j, c_j) \mid j \geq n\}$ is consistent, but for all $i < j$, $\{\varphi'(x; p_i, p'_i, c_i), \varphi(x; p_j, p'_j, c_j)\}$ is inconsistent. This establishes SOP_3. □

For the remainder of the note, fix some $a \in O$ and $b \in G$ such that $E(a, b, 0)$.

Claim 2. $a \mathop{\smile\!\!\!\!\!\!\vert}^{\mathrm{d}}_{\varnothing} b$.

Proof. By the classification of types obtained above, $\mathrm{tp}(a/b)$ is isolated by the formula $\varphi(x, b)\colon (\exists z : C)\, E(x, b, z)$. It suffices to show this formula does not divide over $\varnothing$.

Let $I = (b_i)_{i\in\omega}$ be an indiscernible sequence in $\mathrm{tp}(b/\varnothing)$.

Case 1: $M \models \neg R(b_i, b_j, 0)$ for all $i \neq j$. Then we can find some $a' \in O$ with $M \models E(a', b_i, 0)$, and hence $M \models \varphi(a', b_i)$, for all $i \in \omega$.

Case 2: Otherwise, by indiscernibility, $M \models R(b_i, b_j, 0)$ for all $i \neq j$. Since $R(x, y, 0)$ and $R(x, y, 1)$ are disjoint relations, $M \models \neg R(b_i, b_j, 1)$ for all $i \neq j$. So we can find some $a' \in O$ with $M \models E(a', b_i, 1)$, and hence $M \models \varphi(a', b_i)$, for all $i \in \omega$. $\square$

Claim 3. $a \not\mathop{\smile\!\!\!\!\!\!\vert}^{\mathrm{d}}_{\mathrm{acl}(\varnothing)} b$.

Proof. Note that $\mathrm{acl}(\varnothing) = C$, so $\mathrm{tp}(a/\mathrm{acl}(\varnothing)b)$ contains the formula $E(x, b, 0)$. It suffices to show that this formula divides over C.

By the classification of types obtained above, all elements of G have the same type over C. Let $(b_i)_{i\in\omega}$ be a sequence of elements of G such that for all $i \neq j$, $R(b_i, b_j, 0)$. Then $\{E(x, b, 0) \mid\in \omega\}$ is 2-inconsistent, which witnesses dividing. $\square$

Corollary. *In T, forking is not equal to dividing for complete types.*

Proof. By Claim 2, it suffices to show that $a \not\mathop{\smile\!\!\!\!\!\!\vert}^{\mathrm{f}}_{\varnothing} b$. Suppose for contradiction that $a \mathop{\smile\!\!\!\!\!\!\vert}^{\mathrm{f}}_{\varnothing} b$. By Proposition 2, $\mathrm{acl}(a) \mathop{\smile\!\!\!\!\!\!\vert}^{\mathrm{f}}_{\mathrm{acl}(\varnothing)} \mathrm{acl}(b)$, and by monotonicity, $a \mathop{\smile\!\!\!\!\!\!\vert}^{\mathrm{f}}_{\mathrm{acl}(\varnothing)} b$, so $a \mathop{\smile\!\!\!\!\!\!\vert}^{\mathrm{d}}_{\mathrm{acl}(\varnothing)} b$, contradicting Claim 3. $\square$