Generalized indiscernibles from ultrafilters

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1 Generalized indiscernibles

We fix a complete L-theory T with monster model $\mathbb{M}$ and another (unrelated, but arbitrary) language L'. Our goal in this note is to develop some machinery to produce generalized indiscernibles in $\mathbb{M}$ indexed by L'-structures.

Throughout, let I be an L'-structure which is uniformly locally finite, and let $\mathcal{K} = \mathrm{Age}(I)$. Note that $\mathcal{K}$ is a class of finite L'-structures with the hereditary property (HP) and the joint embedding property (JEP).

We will fix some notation. Given L'-structures A and B, we write $\mathrm{Emb}(A, B)$ for the set of embeddings $A \hookrightarrow B$. We will often deal with L-formula or types in variable context $(x_c)_{c\in C}$, where $C \in \mathcal{K}$. Here we have one tuple of free variables x_c for each element of C. Given an L-formula $\varphi((x_c)_{c\in C})$ and a C-indexed family of tuples $(a_c)_{c\in C}$ from $\mathbb{M}$, we can make sense of $\mathbb{M} \models \varphi((a_c)_{c\in C})$ in the obvious way.

Definition 1.1. Let $\mathcal{I} = (a_i)_{i\in I}$ be a family of tuples from $\mathbb{M}$, indexed by the L'-structure I, and let $D \subseteq \mathbb{M}$ be a small set. We say that $\mathcal{I}$ is a family of ***I*-indexed indiscernibles over** D if for any $C \in \mathcal{K}$ and any embeddings $f, g \in \mathrm{Emb}(C, I)$, we have $\mathrm{tp}_L((a_{f(c)})_{c\in C}/D) = \mathrm{tp}_L((a_{g(c)})_{c\in C}/D)$.

The behavior of a family of I-indexed indiscernibles over D is completely determined by a family of complete types over D, one for each structure in $\mathcal{K}$.

Definition 1.2. A **$\mathcal{K}$-type** (over $D \subseteq \mathbb{M}$) is a family $(p_C)_{C\in\mathcal{K}}$, where each p_C is a partial L-type over D in variable context $(x_c)_{c\in C}$.

Given a $\mathcal{K}$-type $(p_C)_{C\in\mathcal{K}}$ and an L'-structure J with $\mathrm{Age}(J) \subseteq \mathcal{K}$, we define

$$p_J((x_j)_{j\in J}) = \bigcup_{C\in\mathcal{K}} \bigcup_{f\in\mathrm{Emb}(C,J)} p_C((x_{f(c)})_{c\in C}).$$

If $\mathcal{J} = (a_j)_{j\in J}$ realizes p_J, we say that $\mathcal{J}$ is a J-indexed realization of $(p_C)_{C\in\mathcal{K}}$.

We say a $\mathcal{K}$-type $(p_C)_{C\in\mathcal{K}}$ is **consistent** if p_J is consistent for every L'-structure J with $\mathrm{Age}(J) \subseteq \mathcal{K}$. We say $(p_C)_{C\in\mathcal{K}}$ is **complete** if it is consistent and p_C is a complete type over D for all $C \in \mathcal{K}$.

If $\mathrm{Age}(J) \subseteq \mathcal{K}$, then for any finite tuple $\overline{j} = (j_1, \ldots, j_n)$ from J, the L'-substructure of J generated by $\overline{j}$ (call it A) is in $\mathcal{K}$. If the $\mathcal{K}$-type $(p_C)_{C\in\mathcal{K}}$ is complete, then $p_J((x_j)_{j\in J})$ contains the complete type $p_A((x_a)_{a\in A})$, which in particular contains a complete type in the variables $x_{j_1}, \ldots, x_{j_n}$. It follows that $p_J((x_j)_{j\in J})$ is a complete type in the variables $(x_j)_{j\in J}$.

Observe that if $\mathcal{J} = (b_j)_{j\in J}$ is a J-indexed realization of a complete $\mathcal{K}$-type $(p_C)_{C\in\mathcal{K}}$ over D, then $\mathcal{J}$ is a family of J-indexed indiscernibles over D. Indeed, for any $C \in \mathcal{K}$ and any embeddings $f, g \in \mathrm{Emb}(C, J)$, $(b_{f(c)})_{c\in C}$ and $(b_{g(c)})_{c\in C}$ both realize the complete type p_C over D.

So if we want to find generalized indiscernibles, we would like to find (complete) $\mathcal{K}$-types. One way to do this is to read them off from a (not necessarily indiscernible) I-indexed family of tuples $\mathcal{I} = (a_i)_{i\in I}$ from $\mathbb{M}$: an $L(D)$-formula $\varphi((x_c)_{c\in C})$ goes in the type if and only if the set of embeddings $f \in \mathrm{Emb}(C, I)$ such that $\mathbb{M} \models \varphi((a_{f(c)})_{c\in C})$ is "large". Of course, we have to decide what we mean by "large". For now, we encode the "large" sets as an arbitrary family $\mathcal{G}$.

Definition 1.3. Let $\mathcal{G} = (G_C)_{C\in\mathcal{K}}$ be a family of sets with $G_A \subseteq \mathcal{P}(\mathrm{Emb}(A, I))$ for each $A \in \mathcal{K}$. Let $\mathcal{I} = (a_i)_{i\in I}$ be a family of tuples from $\mathbb{M}$, and let $D \subseteq \mathbb{M}$ be a small set. The **$\mathcal{G}$-Ehrenfeucht-Mostowski type** of $\mathcal{I}$ over D, denoted $\mathrm{EM}_{\mathcal{G}}(\mathcal{I}/D)$, is the $\mathcal{K}$-type $(p_C)_{C\in K}$ defined by

$$p_C = \{\varphi((x_c)_{c\in C}) \in L(D) \mid \{f \in \mathrm{Emb}(C, I) \mid \mathbb{M} \models \varphi((x_{f(c)})_{c\in C})\} \in G_C\}.$$

Example 1.4. The classical EM-type over D of a sequence $\mathcal{I} = (a_n)_{n\in\omega}$ is $\mathrm{EM}_{\mathcal{G}}(\mathcal{I}/D)$, where $\mathcal{I}$ is indexed by $I = (\omega, \leq)$ and $\mathcal{G} = (G_C)_{C\in\mathcal{K}}$, where $G_C = \{\mathrm{Emb}(C, I)\}$ for all $C \in \mathcal{K}$. That is, the only set of embeddings which is "large" is the set of all embeddings.

Of course, for an arbitrary set $\mathcal{G}$ and an arbitrary family $\mathcal{I} = (a_i)_{i\in I}$, the $\mathcal{K}$-type $\mathrm{EM}_{\mathcal{G}}(\mathcal{I}/D)$ will typically not be complete (or even consistent). To ensure completeness, we need the sets G_C in $\mathcal{G}$ to be ultrafilters on $\mathrm{Emb}(C, I)$ for all $C \in \mathcal{K}$, and to ensure consistency, we need these ultrafilters to cohere in a precise sense.

2 Age ultrafilter families

Given $A, B \in \mathcal{K}$ and an embedding $f \in \mathrm{Emb}(A, B)$, we obtain a "restriction" map $(- \circ f)\colon \mathrm{Emb}(B, I) \to \mathrm{Emb}(A, I)$. Recall that given a filter F on a X and a function $\rho\colon X \to Y$, we can "push forward" F along ρ, obtaining a filter ρ_*F on Y, defined by

$$\rho_*F = \{Z \subseteq Y \mid \rho^{-1}[Z] \in F\}.$$

We write βX for the Stone space of ultrafilters on X. If U is an ultrafilter on X, then ρ_*U is an ultrafilter on Y, so ρ_* is a map $\beta X \to \beta Y$.

In particular, for every embedding $f \in \mathrm{Emb}(A, B)$, we obtain a "push forward along the restriction" map $(- \circ f)_*$ which maps (ultra)filters on $\mathrm{Emb}(B, I)$ to (ultra)filters on $\mathrm{Emb}(A, I)$.

Definition 2.1. An **age filter family** on I is a family $\mathcal{F} = (F_A)_{A\in\mathcal{K}}$ such that:

(1) For each $A \in \mathcal{K}$, F_A is a filter on $\mathrm{Emb}(A, I)$.

(2) For each embedding $f\colon A \hookrightarrow B$ with $A, B \in \mathcal{K}$, $(- \circ f)_*(F_B) = F_A$.

We call (2) the **pushforward condition**.

The age filter family $\mathcal{F}$ is **proper** if each F_A is a proper filter. It is an **age ultrafilter family** if each F_A is an ultrafilter.

We say an age filter family $\mathcal{F}' = (F'_A)_{A\in\mathcal{K}}$ **extends** $\mathcal{F}$ if $F_A \subseteq F'_A$ for all $A \in \mathcal{K}$.

Lemma 2.2. *Let $\mathcal{I} = (a_i)_{i\in I}$ be a family of tuples from $\mathbb{M}$, and let $D \subseteq \mathbb{M}$ be a small set.*

(1) If $\mathcal{G}$ extends to a proper age filter family on I, then $\mathrm{EM}_{\mathcal{G}}(\mathcal{I}/D)$ is a consistent $\mathcal{K}$-type.

(2) If $\mathcal{U}$ is an age ultrafilter family on I, then $\mathrm{EM}_{\mathcal{U}}(\mathcal{I}/D)$ is a complete $\mathcal{K}$-type.

Proof. For (1), let $\mathcal{F} = (F_C)_{C\in\mathcal{K}}$ be an age filter family on $I = (a_i)_{i\in I}$ extending $\mathcal{G}$. It suffices to show that $\mathrm{EM}_{\mathcal{F}}(\mathcal{I}/D) = (p_C)_{C\in\mathcal{K}}$ is a consistent $\mathcal{K}$-type. Let J be an L'-structure with $\mathrm{Age}(J) \subseteq \mathcal{K}$. We would like to show by compactness that $p_J((x_j)_{j\in J})$ is consistent.

A finite subset of p_J has the form $\{\varphi_i((x_{f_i(c)})_{c\in C_i}) \mid 1 \leq i \leq n\}$, where for each i, $C_i \in \mathcal{K}$, $f_i \in \mathrm{Emb}(C_i, J)$, and $\varphi_i((x_c)_{c\in C_i}) \in p_{C_i}$. Let B be the substructure of J generated by $\bigcup_{i=1}^n f_i[C_i]$. Then we can view each f_i as an embedding $C_i \hookrightarrow B$. Let $\widehat{\varphi}_i((x_b)_{b\in B})$ be the formula obtained from φ_i by replacing each tuple of variables x_c by $x_{f_i(c)}$ and adding dummy variables x_b for all $b \in B \setminus f[C_i]$. It suffices to show that $\{\widehat{\varphi}_i((x_b)_{b\in B}) \mid 1 \leq i \leq n\}$ is consistent.

Since $\varphi_i((x_c)_{c\in C_i}) \in p_{C_i}$, we have

$$X_i = \{g \in \mathrm{Emb}(C_i, I) \mid \mathbb{M} \models \varphi_i((a_{g(c)})_{c\in C_i}\} \in F_{C_i},$$

so the preimage of this set under $(- \circ f_i)$ is in F_B. That is:

$$\begin{aligned}(- \circ f_i)^{-1}[X_i] &= \{h \in \mathrm{Emb}(B, I) \mid \mathbb{M} \models \varphi_i((a_{h(f_i(c))})_{c\in C_i})\}\\ &= \{h \in \mathrm{Emb}(B, I) \mid \mathbb{M} \models \widehat{\varphi}_i((a_{h(b)})_{b\in B})\} \in F_B.\end{aligned}$$

Now since F_B is a proper filter, we can find some embedding $h \in \mathrm{Emb}(B, I)$ with $h \in \bigcap_{i=1}^n (- \circ f_i)^{-1}[X_i]$. Then we have $\mathbb{M} \models \bigwedge_{i=1}^n \widehat{\varphi}_i((a_{h(b)})_{b\in B})$, so this set of formulas is consistent, as desired.

For (2), since $\mathcal{U} = (U_C)_{C\in\mathcal{K}}$ is an age filter family on I, $\mathrm{EM}_{\mathcal{U}}(\mathcal{I}/D) = (p_C)_{C\in\mathcal{K}}$ is a consistent $\mathcal{K}$-type by (1). It remains to show that each p_C is a complete type. So let $\varphi((x_c)_{c\in C})$ be a formula. The sets

$$\{f \in \mathrm{Emb}(C, I) \mid \mathbb{M} \models \varphi((a_{f(c)})_{c\in C}\}$$

and

$$\{f \in \mathrm{Emb}(C, I) \mid \mathbb{M} \models \neg\varphi((a_{f(c)})_{c\in C}\}$$

are complementary, so one of them is in U_C, and hence either $\varphi((x_c)_{c\in C})$ or $\neg\varphi((x_c)_{c\in C})$ is in p_C. □

Example 2.3. Let $I = (\mathbb{N}, \leq)$. Then $\mathcal{K} = \mathrm{Age}(I)$ is the class of all finite linear orders. For any linear order A with n elements, the set $\mathrm{Emb}(A, I)$ can be identified with the set $[\mathbb{N}]^n$ of strictly increasing n-tuples from $\mathbb{N}$. So an age ultrafilter family on I is essentially a family $(U_n)_{n\in\mathbb{N}}$, where U_n is an ultrafilter on $[\mathbb{N}]^n$, satisfying the pushforward condition.

Given such a family $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$, U_1 is just an ultrafilter on $\mathbb{N}$, which must be non-principal. Indeed, suppose that for some $a \in \mathbb{N}$, $\{a\} \in U_1$. Let f_1 and f_2 be the two embeddings of the one-element linear order into the two-element linear order, where f_1 sends the unique element to the first element and f_2 sends to the unique element to the second element. Since $(- \circ f_1)_*U_2 = U_1$, we have $(- \circ f_1)^{-1}[\{a\}] = \{(a, b) \mid a < b\} \in U_2$, and since $(- \circ f_2)_*U_2 = U_1$, we have $(-\circ f_2)^{-1}[\{a\}] = \{(b, a) \mid b < a\} \in U_2$. But $(-\circ f_1)^{-1}[\{a\}]\cap(-\circ f_2)^{-1}[\{a\}] = \varnothing$, contradicting the fact that U_2 is an ultrafilter.

On the other hand, any non-principal ultrafilter U on $\mathbb{N}$ can be canonically extended to an age ultrafilter family on I. Recall that given ultrafilters U and V on X and Y, respectively, we define an ultrafilter $U \otimes V$ on $X \times Y$ by

$$Z \in U \otimes V \iff \{a \in X \mid \{b \in Y \mid (a, b) \in Z\} \in V\} \in U.$$

This operation on ultrafilters is associative, so we have a well-defined ultrafilter $U^{\otimes n}$ on $\mathbb{N}^n$. An easy induction shows that $[\mathbb{N}]^n \in U^{\otimes n}$, so $U^{\otimes n}$ restricts to an ultrafilter on this set. Defining $U_n = U^{\otimes n}|_{[\mathbb{N}]^n}$ gives an age ultrafilter family with $U_1 = U$. It is a bit tedious, but not hard, to check the pushforward condition.

Letting $\mathcal{U}$ be the age ultrafilter family defined above from the ultrafilter U, and given any sequence $\mathcal{I}$ of tuples from $\mathbb{M}$, a realization of $\mathrm{EM}_{\mathcal{U}}(\mathcal{I}/D)$ is a Morley sequence in the global D-invariant type $\mathrm{Av}(\mathcal{I}; U)$.

Which other structures admit age ultrafilter families? The answer to this question turns out to depend only on the age: we will prove that an age ultrafilter family on I exists if and only if $\mathrm{Age}(I)$ is a Ramsey class.

3 Ramsey classes

Definition 3.1. For structures A, B, and C, and $k \in \mathbb{N}$, the notation

$$C \to (B)^A_k$$

means that for every coloring of $\mathrm{Emb}(A, C)$ by k colors (i.e., every function $\gamma\colon \mathrm{Emb}(A, C) \to P$, where $|P| = k$), there is some embedding $f\colon B \hookrightarrow C$ such that the set of embeddings $A \hookrightarrow C$ which factor through f are monochromatic (i.e., there exists $c \in P$ such that for all $g\colon A \hookrightarrow B$, $\gamma(f \circ g) = c$).

We say that $\mathcal{K}$ is a **Ramsey class** if for all $A, B \in \mathcal{K}$ and all $k \in \mathbb{N}$, there exists $C \in \mathcal{K}$ such that $C \to (B)^A_k$.

We will now collect a few basic properties of Ramsey classes.

Lemma 3.2. *Every Ramsey class has the amalgamation property (AP).*

For the following lemma, we extend the arrow notation. The notation

$$C \to (B)^{A_1,\dots,A_n}_{k_1,\dots,k_n}$$

means that given a coloring of $\mathrm{Emb}(A_i, C)$ by k_i colors for each i, there is some embedding $f\colon B \hookrightarrow C$ such that for each i, the set of embeddings $A_i \hookrightarrow C$ which factor through f are monochromatic.

Lemma 3.3. *Suppose $\mathcal{K}$ is a Ramsey class. Then for all $A_1, \dots, A_n, B \in \mathcal{K}$ and all $k_1, \dots, k_n \in \mathbb{N}$, there exists $C \in \mathcal{K}$ such that $C \to (B)^{A_1,\dots,A_n}_{k_1,\dots,k_n}$.*

Lemma 3.4. *$\mathcal{K} = \mathrm{Age}(I)$ is a Ramsey class if and only if for all $A, B \in \mathcal{K}$ and $k \in \mathbb{N}$, $I \to (B)^A_k$.*

Now we can prove one half of our desired characterization of the existence of age ultrafilter families.

Theorem 3.5. *Suppose I admits an age ultrafilter family $\mathcal{U} = (U_A)_{A \in \mathcal{K}}$. Then $\mathcal{K} = \mathrm{Age}(I)$ is a Ramsey class.*

Proof. Fix $A, B \in \mathcal{K}$ and $k \in \mathbb{N}$. By Lemma 3.4, it suffices to show $I \to (B)^A_k$. So let $\gamma\colon \mathrm{Emb}(A, I) \to P$ be a coloring with $|P| = k$. For each $c \in P$, define

$$X_c = \gamma^{-1}[\{c\}] \subseteq \mathrm{Emb}(A, I).$$

Then the X_c form a finite partition of $\mathrm{Emb}(A, I)$, so $X_c \in U_A$ for some $c \in P$.

For each embedding $f \in \mathrm{Emb}(A, B)$, we have $(- \circ f)_* U_B = U_A$, so

$$(- \circ f)^{-1}(X_c) = \{g \in \mathrm{Emb}(B, I) \mid \gamma(g \circ f) = c\} \in U_B.$$

Since $\mathrm{Emb}(A, B)$ is finite (as B is finite) and U_B is an ultrafilter, the intersection $\bigcap_{f \in \mathrm{Emb}(A,B)} (- \circ f)^{-1}(X_c)$ is nonempty. Let $g \in \mathrm{Emb}(B, I)$ be in the intersection. Then for all $f \in \mathrm{Emb}(A, B)$, $\gamma(g \circ f) = c$, as desired. □

4 Existence of age ultrafilter families

Our goal in this section is to prove the converse of Theorem 3.5.

Theorem 4.1. *Suppose $\mathcal{K} = \mathrm{Age}(I)$ is a Ramsey class. Then I admits an age ultrafilter family.*

We actually give two proofs. The first proof proceeds directly by compactness. It is shorter, but it seems (at least to me) to be less informative. The strategy of the second proof is to show, by Zorn's Lemma, that every proper age filter family extends to an age ultrafilter family. This may give more control over the construction, by allowing us to encode some constraints into a proper age filter family before running the nonconstructive part of the argument.

First proof of Theorem 4.1. We consider the compact space $\prod_{C\in\mathcal{K}} \beta\mathrm{Emb}(C, I)$. For each embedding $f\colon A \hookrightarrow B$ and each set $X \subseteq \mathrm{Emb}(A, I)$, define

$$V_{f,X} = \{(U_C)_{C\in\mathcal{K}} \mid X \in U_A \text{ iff } (-\circ f)^{-1}(X) \in U_B\}.$$

Each set $V_{f,X}$ is closed, and the set of age ultrafilter families on I is equal to the intersection of all sets of the form $V_{f,X}$. So by compactness, it suffices to show that any finitely many of these sets have non-empty intersection.

To this end, we define another family of subsets of $\prod_{C\in\mathcal{K}} \beta\mathrm{Emb}(C, I)$. Let $\mathcal{A}$ be a finite set of structures in $\mathcal{K}$, B a structure in $\mathcal{K}$, and $\mathcal{X} = (\mathcal{X}_A)_{A\in\mathcal{A}}$ a family such that for each $A \in \mathcal{A}$, $\mathcal{X}_A$ is a set of finitely many subsets of $\mathrm{Emb}(A, I)$. Then we define

$$\begin{aligned} V_{\mathcal{A},B,\mathcal{X}} = \{(U_C)_{C\in\mathcal{K}} \mid &\text{for all } A \in \mathcal{A}, \text{ all } f\colon A \hookrightarrow B, \text{ and all } X \in \mathcal{X}_A, \\ &X \in U_A \text{ iff } (-\circ f)^{-1}[X] \in U_B\}. \end{aligned}$$

We first show that the intersection of any finitely many sets of the form $V_{f,X}$ contains a set of the form $V_{\mathcal{A},B,\mathcal{X}}$. Then we use Lemma 3.3 to show that any set of the form $V_{\mathcal{A},B,\mathcal{X}}$ is non-empty.

So fix finitely many sets $V_{f_1,X_1}, \dots V_{f_n,X_n}$, where for each i, $f_i\colon A_i \hookrightarrow B_i$ is an embedding and $X_i \subseteq \mathrm{Emb}(A_i, I)$. By JEP, we can pick a structure $B \in \mathcal{K}$ and embeddings $g_i\colon B_i \hookrightarrow B$ for each i.

Let $\mathcal{A} = \{A_1, \dots, A_n, B_1, \dots, B_n\}$, and for each $A \in \mathcal{A}$, let

$$\mathcal{X}_A = \{X_i \mid A = A_i\} \cup \{(-\circ f_i)^{-1}[X_i] \mid A = B_i\}.$$

Note that since the same structure A may occur multiple times among the A_i and the B_i, $\mathcal{X}_A$ may have more than one element. Let $\mathcal{X} = (\mathcal{X}_A)_{A\in\mathcal{A}}$.

We claim that $V_{\mathcal{A},B,\mathcal{X}} \subseteq \bigcap_{i=1}^n V_{f_i,X_i}$. So let $(U_C)_{C\in\mathcal{K}} \in V_{\mathcal{A},B,\mathcal{X}}$. Then for each $1 \le i \le n$, we have A_i and B_i in $\mathcal{A}$, $X_i \in \mathcal{X}_{A_i}$ and $(-\circ f_i)^{-1}[X_i] \in \mathcal{X}_{B_i}$, so

$$\begin{aligned} X_i \in U_{A_i} &\text{ iff } (-\circ (g_i \circ f_i))^{-1}[X_i] \in U_B \\ &\text{ iff } (-\circ g_i)^{-1}[(-\circ f_i)^{-1}[X_i]] \in U_B \\ &\text{ iff } (-\circ f_i)^{-1}[X_i] \in U_{B_i} \end{aligned}$$

So $(U_C)_{C\in\mathcal{K}} \in V_{f_i,X_i}$, as desired.

It remains to show that each set of the form $V_{\mathcal{A},B,\mathcal{X}}$ is non-empty. Enumerate $\mathcal{A}$ as $A_1, \dots, A_n$, and for each i, let $k_i = 2^{|\mathcal{X}_{A_i}|}$. By Lemma 3.3, we can find $C \in \mathcal{K}$ such that $C \to (B)^{A_1,\dots,A_n}_{k_1,\dots,k_n}$. Since $C \in \mathrm{Age}(I)$, we can pick an embedding $h\colon C \to I$. Now for each $1 \le i \le n$, we define $\gamma_i\colon \mathrm{Emb}(A_i, C) \to \mathcal{P}(\mathcal{X}_{A_i})$ by

$$\gamma_i(e) = \{X \in \mathcal{X}_{A_i} \mid h \circ e \in X\}.$$

By our coloring property, there is an embedding $g\colon B \hookrightarrow C$ such that for each $1 \le i \le n$, there exists $\mathcal{X}'_{A_i} \subseteq \mathcal{X}_{A_i}$ such that $\gamma_i(g \circ f) = \mathcal{X}'_{A_i}$ for all $f\colon A_i \hookrightarrow B$.

Let U_B be the principal ultrafilter generated by $\{h \circ g\}$. For each $1 \le i \le n$, we define U_{A_i} and check that for all $f \in \mathrm{Emb}(A, B)$ and all $X \in \mathcal{X}_{A_i}$, $X \in U_{A_i}$ if and only if $(- \circ f)^{-1}[X] \in U_B$. Having done this, we can define U_C to be an arbitrary ultrafilter on $\mathrm{Emb}(C, I)$ for all $C \in \mathcal{K} \setminus \mathcal{A}$, and we will have $(U_C)_{C\in\mathcal{K}} \in V_{\mathcal{A},B,\mathcal{X}}$.

Given i, if there is no embedding $A_i \hookrightarrow B$, then the condition is vacuously satisfied, and we can pick U_{A_i} arbitrarily. Otherwise, pick some $f \in \mathrm{Emb}(A_i, B)$ and let U_{A_i} be the principal ultrafilter generated by $\{h \circ g \circ f\}$ (if $A_i = B$, we can pick f to be the identity, so this agrees with the choice of U_B given above). Now for any $f' \in \mathrm{Emb}(A, B)$ and any $X \in \mathcal{X}_{A_i}$, we have

$$\begin{aligned} X \in U_A &\text{ iff } h \circ g \circ f \in X \\ &\text{ iff } X \in \gamma_i(g \circ f) \\ &\text{ iff } X \in \mathcal{X}'_{A_i} \\ &\text{ iff } X \in \gamma_i(g \circ f') \\ &\text{ iff } h \circ g \circ f' \in X \\ &\text{ iff } h \circ g \in (- \circ f')^{-1}[X] \\ &\text{ iff } (- \circ f')^{-1}[X] \in U_B. \end{aligned}$$ □

We now embark on the second proof. Our first task is to understand when I admits a proper age filter family. It turns out that this property also depends only on the age: a proper age filter family on I exists if and only if $\mathrm{Age}(I)$ is a Fraïssé class (since $\mathrm{Age}(I)$ always has HP and JEP, this amounts to saying that $\mathrm{Age}(I)$ has AP). This fact is rather striking in conjunction with Theorems 3.5 and 4.1. From this perspective, Lemma 3.2 is not an accident – the Ramsey property is a natural strengthening of AP.

Given an embedding $f\colon A \hookrightarrow B$, we define

$$E_f = \{h \in \mathrm{Emb}(A, I) \mid \text{there exists } g \in \mathrm{Emb}(B, I) \text{ s.t. } g \circ f = h\},$$

the set of embeddings $A \hookrightarrow I$ which extend along f. Equivalently, E_f is the image of the map $(- \circ f)\colon \mathrm{Emb}(B, I) \to \mathrm{Emb}(A, I)$. Note that if $\mathcal{F} = (F_A)_{A\in\mathcal{K}}$ is an age filter family on I, then we have $(- \circ f)^{-1}(E_f) = \mathrm{Emb}(B, I) \in F_B$, so $E_f \in F_A$.

Lemma 4.2. *If I admits a proper age filter family, then $\mathcal{K} = \mathrm{Age}(I)$ has AP.*

Proof. Let $\mathcal{F} = (F_A)_{A\in\mathcal{K}}$ be a proper age filter family on I. Suppose $f_1\colon A \hookrightarrow B_1$ and $f_2\colon A \hookrightarrow B_2$ are embeddings with $A, B_1, B_2 \in \mathcal{K}$. By the observation above, we have $E_{f_1} \cap E_{f_2} \in F_A$, so since F_A is proper, there is some embedding $h\colon A \to I$ which extends along both f_1 and f_2. That is, there are embeddings $g_1\colon B_1 \hookrightarrow I$ and $g_2\colon B_2 \hookrightarrow I$ such that $g_1 \circ f_1 = h = g_2 \circ f_2$. Taking C to be the substructure of I generated by $g_1(B_1) \cup g_2(B_2)$, we have $C \in \mathcal{K}$. Then g_1 and g_2 restrict to embeddings $B_1 \hookrightarrow C$ and $B_2 \hookrightarrow C$ which witness AP. □

Our next goal is to understand the age filter family generated by a family of subsets of $\mathrm{Emb}(A, I)$ for various $A \in \mathcal{K}$. An age filter family is closed under supersets, intersections, and preimages and images under maps of the form $(- \circ f)$. When $\mathcal{K}$ has AP, it turns out that any set obtained by some composition of these operations on a family of generating sets can be given a kind of "normal form": it is a superset of an image of an intersection of preimages of generating sets. This normal form allows us to understand age filter families much more concretely. And in light of Lemma 4.2, we lose nothing by assume $\mathcal{K}$ has AP.

Lemma 4.3. *Assume $\mathcal{K}$ has AP. Let $\mathcal{G} = (G_A)_{A \in \mathcal{K}}$ be a family of sets with $G_A \subseteq \mathcal{P}(\mathrm{Emb}(A, I))$ for each $A \in \mathcal{K}$. For each $A \in K$, let F_A consist of all sets Y such that*

$$(- \circ g)\left[\bigcap_{i=1}^{n} (- \circ f_i)^{-1}[X_i]\right] \subseteq Y$$

where $g \in \mathrm{Emb}(A, B)$ for some $B \in \mathcal{K}$, and for all $1 \leq i \leq n$, $f_i \in \mathrm{Emb}(C_i, B)$ and $X_i \in G_{C_i}$ for some $C_i \in \mathcal{K}$. Note that we allow $n = 0$, in which case the intersection $\bigcap_{i=1}^{n} (- \circ f_i)^{-1}[X_i]$ is the whole set $\mathrm{Emb}(B, I)$. Then $\mathcal{F} = (F_A)_{A \in \mathcal{K}}$ is the minimal age filter family such that $G_A \subseteq F_A$ for all $A \in \mathcal{K}$.

Proof. We begin by observing that

$$(- \circ g)\left[\bigcap_{i=1}^{n} (- \circ f_i)^{-1}[X_i]\right] \subseteq Y \text{ iff } \bigcap_{i=1}^{n} (- \circ f_i)^{-1}[X_i] \subseteq (- \circ g)^{-1}[Y].$$

It will often be more convenient to work with the second characterization.

First, we check that $G_A \subseteq F_A$ for all $A \in \mathcal{K}$. For any $X \in G_A$, we can take $n = 1$, $B = C_1 = A$, and $g = f_1 = \mathrm{id}_A$. Then

$$\bigcap_{i=1}^{1} (- \circ \mathrm{id}_A)^{-1}[X] \subseteq (- \circ \mathrm{id}_A)^{-1}[X],$$

so $X \in F_A$.

Next, we check that $\mathcal{F}$ is an age filter family on I. First, we fix $A \in \mathcal{K}$ and check that F_A is a filter on $\mathrm{Emb}(A, I)$. It is clearly closed under superset, and it contains $\mathrm{Emb}(A, I)$, witnessed by taking $n = 0$ and $g = \mathrm{id}_A$.

Suppose $Y, Y' \in F_A$. Then we have

$$\bigcap_{i=1}^{n} (- \circ f_i)^{-1}[X_i] \subseteq (- \circ g)^{-1}[Y], \text{ and}$$
$$\bigcap_{j=1}^{n'} (- \circ f'_j)^{-1}[X'_j] \subseteq (- \circ g')^{-1}[Y'],$$

where $g \in \mathrm{Emb}(A, B)$ and $g' \in \mathrm{Emb}(A, B')$ for some $B, B' \in \mathcal{K}$, for all $1 \leq i \leq n$, $f_i \in \mathrm{Emb}(C_i, B)$ and $X_i \in G_{C_i}$ for some $C_i \in \mathcal{K}$, and for all $1 \leq j \leq n'$, $f'_j \in \mathrm{Emb}(C'_i, B')$ and $X'_i \in G_{C'_i}$ for some $C'_i \in \mathcal{K}$.

By AP, pick some $B^* \in K$ and embeddings $h\colon B \hookrightarrow B^*$ and $h'\colon B' \hookrightarrow B^*$ such that $h \circ g = h' \circ g'$. Call this embedding g^*. Then we have

$$\begin{aligned}(- \circ g^*)^{-1}[Y \cap Y'] &= (- \circ g^*)^{-1}[Y] \cap (- \circ g^*)^{-1}[Y'] \\ &= (- \circ h)^{-1}[(- \circ g)^{-1}[Y]] \cap (- \circ h')^{-1}[(- \circ g')^{-1}[Y']] \\ &\supseteq (- \circ h)^{-1}\left[\bigcap_{i=1}^{n}(- \circ f_i)^{-1}[X_i]\right] \cap (- \circ h')^{-1}\left[\bigcap_{j=1}^{n'}(- \circ f'_j)^{-1}[X'_j]\right] \\ &= \bigcap_{i=1}^{n}(- \circ h)^{-1}[(- \circ f_i)^{-1}[X_i]] \cap \bigcap_{j=1}^{n'}(- \circ h')^{-1}[(- \circ f'_i)^{-1}[X'_i]] \\ &= \bigcap_{i=1}^{n}(- \circ h \circ f_i)^{-1}[X_i] \cap \bigcap_{j=1}^{n'}(- \circ h' \circ f'_i)^{-1}[X'_i],\end{aligned}$$

so $Y \cap Y' \in F_A$.

It remains to check the pushforward condition. Fix $f \in \mathrm{Emb}(A, A')$ and $Y \subseteq \mathrm{Emb}(A, I)$. We would like to show that $Y \in F_A$ if and only if $(- \circ f)^{-1}[Y] \in F_{A'}$.

Suppose $(- \circ f)^{-1}[Y] \in F_{A'}$. This is witnessed by

$$\bigcap_{i=1}^{n}(- \circ f_i)^{-1}[X_i] \subseteq (- \circ g)^{-1}[(- \circ f)^{-1}[Y]],$$

where $g \in \mathrm{Emb}(A', B)$ for some $B \in \mathcal{K}$, and for all $1 \leq i \leq n$, $f_i \in \mathrm{Emb}(C_i, B)$ and $X_i \in G_{C_i}$ for some $C_i \in \mathcal{K}$. But this just says

$$\bigcap_{i=1}^{n}(- \circ f_i)^{-1}[X_i] \subseteq (- \circ g \circ f)^{-1}[Y],$$

which also witnesses $Y \in F_A$.

Finally, suppose $Y \in F_A$, witnessed by

$$\bigcap_{i=1}^{n}(- \circ f_i)^{-1}[X_i] \subseteq (- \circ g)^{-1}[Y],$$

where $g \in \mathrm{Emb}(A, B)$ for some $B \in \mathcal{K}$, and for all $1 \leq i \leq n$, $f_i \in \mathrm{Emb}(C_i, B)$ and $X_i \in G_{C_i}$ for some $C_i \in \mathcal{K}$.

By AP, pick some $B' \in K$ and embeddings $f'\colon A' \hookrightarrow B'$ and $g'\colon B \hookrightarrow B'$

such that $f' \circ f = g' \circ g$. We have

$$\begin{aligned}(- \circ f')^{-1}[(- \circ f)^{-1}[Y]] &= (- \circ g')^{-1}[(- \circ g)^{-1}[Y]] \\ &\supseteq (- \circ g')^{-1}\left[\bigcap_{i=1}^{n}(- \circ f_i)^{-1}[X_i]\right] \\ &= \bigcap_{i=1}^{n}(- \circ g')^{-1}[(- \circ f_i)^{-1}[X_i]] \\ &= \bigcap_{i=1}^{n}(- \circ g' \circ f_i)^{-1}[X_i],\end{aligned}$$

so $(- \circ f)^{-1}[Y] \in F_{A'}$.

This completes the verification that $\mathcal{F}$ is an age filter family. It remains to show that $\mathcal{F}$ is minimal. So suppose $\mathcal{F}' = (F'_A)_{A\in\mathcal{K}}$ is another age filter family such that $G_A \subseteq F'_A$ for all $A \in \mathcal{K}$. Let $Y \in F_A$, witnessed by

$$\bigcap_{i=1}^{n}(- \circ f_i)^{-1}[X_i] \subseteq (- \circ g)^{-1}[Y],$$

where $g \in \mathrm{Emb}(A, B)$ for some $B \in \mathcal{K}$, and for all $1 \le i \le n$, $f_i \in \mathrm{Emb}(C_i, B)$ and $X_i \in G_{C_i}$ for some $C_i \in \mathcal{K}$.

For all $1 \le i \le n$, $X_i \in F'_{C_i}$, so $(- \circ f_i)^{-1}[X_i] \in F'_B$, since $(f_i)_* F'_B = F'_{C_i}$. Since F'_B is a filter, $\bigcap_{i=1}^{n}(- \circ f_i)^{-1}[X_i] \in F'_B$, and also $(- \circ g)^{-1}[Y] \in F'_B$ by our assumption on Y. But then $Y \in F'_A$, since $g_* F'_B = F'_A$. Thus, $F_A \subseteq F'_A$. $\square$

Note that if we take $G_A = \emptyset$ for all $A \in \mathcal{K}$ in Lemma 4.3, we will generate the minimal age filter family $\mathcal{F}^{\min} = (F_A^{\min})_{A\in\mathcal{K}}$ on I. This has a simple description:

$$F_A^{\min} = \{Y \subseteq \mathrm{Emb}(A, I) \mid E_f \subseteq Y \text{ for some } B \in \mathcal{K}, f \in \mathrm{Emb}(A, B)\}.$$

Indeed, since there are no generating sets, we must take $n = 0$ in the description of the sets in $F_A^{\min}$ from Lemma 4.3. We are left with $Y \in F_A^{\min}$ if and only if $(- \circ f)[\mathrm{Emb}(B, I)] = E_f \subseteq Y$ for some $B \in \mathcal{K}$ and $f \in \mathrm{Emb}(A, B)$. Note that this description only holds when $\mathcal{K}$ has AP. Of course, when $\mathcal{K}$ fails to have AP, the description is also simple: $F_A^{\min}$ is the improper filter for all $A \in \mathcal{K}$.

If I is the Fraïssé limit of $\mathcal{K}$, then for all $f\colon A \hookrightarrow B$, every embedding $A \hookrightarrow I$ extends along f, so $E_f = \mathrm{Emb}(A, I)$. Thus $F_A^{\min} = \{\mathrm{Emb}(A, I)\}$ for all $A \in K$. But for less homogeneous I, $\mathcal{F}^{\min}$ is less trivial.

Using Lemma 4.3, we can now prove the converse of Lemma 4.2 and characterize when a family of sets generates a proper age filter family. The key condition is the following generalization of the finite intersection property.

Definition 4.4. Let $\mathcal{G} = (G_A)_{A\in\mathcal{K}}$ be a family of sets with $G_A \subseteq \mathcal{P}(\mathrm{Emb}(A, I))$ for each $A \in \mathcal{K}$. We say that $\mathcal{G}$ has the **preimage finite intersection property (PFIP)** if no finite intersection of preimages of sets in $\mathcal{G}$ is empty. That is, for all $A_1, \ldots, A_n, B \in \mathcal{K}$, embeddings $f_i \colon A_i \hookrightarrow B$ for all i, and sets $X_1, \ldots, X_n$ with $X_i \in G_{A_i}$ for all i, we have

$$\bigcap_{i=1}^{n} (- \circ f_i)^{-1}[X_i] \neq \emptyset.$$

Corollary 4.5. *Suppose $\mathcal{K} = \mathrm{Age}(I)$ has AP. If $\mathcal{G} = (G_A)_{A \in \mathcal{K}}$ has PFIP, then the age filter family generated by $\mathcal{G}$ is proper. In particular, the age filter family $\mathcal{F}^{\min}$ described above is proper.*

Proof. Let $\mathcal{F}$ be the age filter family generated by $\mathcal{G}$. By Lemma 4.3, for any $A \in \mathcal{K}$ and $Y \in F_A$, we have

$$(- \circ g)\left[\bigcap_{i=1}^{n} (- \circ f_i)^{-1}[X_i]\right] \subseteq Y$$

where $g \in \mathrm{Emb}(A, B)$ for some $B \in \mathcal{K}$, and for all $1 \leq i \leq n$, $f_i \in \mathrm{Emb}(C_i, B)$ and $X_i \in G_{C_i}$ for some $C_i \in \mathcal{K}$. Since $\mathcal{G}$ has PFIP, $\bigcap_{i=1}^{n} (- \circ f_i)^{-1}[X_i]$ is non-empty. Since the image of a non-empty set is non-empty, Y is non-empty. So $\mathcal{F}$ is proper.

In particular, if we take $G_A = \emptyset$ for all $A \in \mathcal{K}$, $\mathcal{G}$ trivially has the PFIP (since each set $\mathrm{Emb}(A, I)$ is non-empty for $A \in \mathcal{K}$). So the age filter family $\mathcal{F}^{\min}$ generated by $\mathcal{G}$ is proper. □

At this point, we are set up to run a standard Zorn's Lemma argument for the existence of age ultrafilter families. The key remaining step is to show that given a proper age filter family $\mathcal{F}$ and a set $X \subseteq \mathrm{Emb}(A, I)$, we can add X or its complement to $\mathcal{F}$, and the resulting family will still have PFIP. Unlike the classical case of extending a filter, we cannot always choose arbitrarily whether to add X or its complement. To check PFIP, we need to use the Ramsey property, as well as the fact that all sets in $\mathcal{F}$ are "thick". This terminology comes from topological dynamics, and its use in this context is due to Zucker.

Definition 4.6. Let $A \in \mathcal{K}$ and $X \subseteq \mathrm{Emb}(A, I)$. We say X is **thick** if for all $B \in \mathcal{K}$, there exists an embedding $g \colon B \hookrightarrow I$ such that

$$\{g \circ f \mid f \in \mathrm{Emb}(A, B)\} \subseteq X.$$

Lemma 4.7. *Suppose $\mathcal{F}$ is a proper age filter family on I. Then every set in $\mathcal{F}$ is thick.*

Proof. Fix $A \in \mathcal{K}$ and $X \in F_A$. We would like to show X is thick. So pick $B \in \mathcal{K}$. For each embedding $f \colon A \hookrightarrow B$, we have $(- \circ f)^{-1}[X] \in F_B$. Since F_B is a proper filter,

$$\bigcap_{f \in \mathrm{Emb}(A,B)} (- \circ f)^{-1}[X] \neq \emptyset.$$

Picking some g in the intersection, we have $g \circ f \in X$ for all $f \in \mathrm{Emb}(A, B)$, as desired. □

Lemma 4.8. *Assume $\mathcal{K}$ is a Ramsey clasas. Let $\mathcal{F} = (F_C)_{C\in\mathcal{K}}$ be a proper age filter family on I, let $X \subseteq \mathrm{Emb}(A, I)$, and let $X' = \mathrm{Emb}(A, I) \setminus X$ be its complement. Then there is a proper age filter family $\mathcal{F}' = (F'_C)_{C\in\mathcal{K}}$ extending $\mathcal{F}$ such that either $X \in F'_A$ or $X' \in F'_A$.*

Proof. Define $\mathcal{G} = (G_C)_{C\in\mathcal{K}}$ and $\mathcal{G}' = (G'_C)_{C\in\mathcal{K}}$ by $G_A = F_A \cup \{X\}$, $G'_A = F_A \cup \{X'\}$, and $G_C = G'_C = F_C$ for all $C \neq A$. By Corollary 4.5, it suffices to show that either $\mathcal{G}$ or $\mathcal{G}'$ has PFIP. So suppose for contradiction that neither has PFIP.

In the case of $\mathcal{G}$, this means that there is some $B \in \mathcal{K}$ and finitely many subsets of $\mathrm{Emb}(B, I)$, each of which is a preimage of a set in $\mathcal{G}$ under a map of the form $(- \circ f)$, with empty intersection. Any set in $\mathcal{G}$ other than X is already in $\mathcal{F}$, and since $\mathcal{F}$ is an age filter family, it is closed under preimages and finite intersection. So we may assume that there are finitely many embeddings $f_1, \ldots, f_n \colon A \hookrightarrow B$ and a set $Y \in F_B$ such that

$$Y \cap \bigcap_{i=1}^{n} (- \circ f_i)^{-1}[X] = \emptyset.$$

Similarly, in the case of $\mathcal{G}'$, there is some $B' \in \mathcal{K}$, finitely many embeddings $f'_1, \ldots, f'_m \colon A \hookrightarrow B'$, and a set $Y' \in F_{B'}$ such that

$$Y' \cap \bigcap_{j=1}^{m} (- \circ f'_j)^{-1}[X] = \emptyset.$$

Since $\mathcal{K}$ has JEP, we can pick $B^* \in \mathcal{K}$ and embeddings $g \colon B \hookrightarrow B^*$ and $g' \colon B' \hookrightarrow B^*$. Let $Z = (- \circ g)^{-1}[Y] \cap (- \circ g')^{-1}[Y']$. Since $\mathcal{F}$ is an age filter family, $Z \in F_{B^*}$. And we have

$$\begin{aligned} Z \cap \bigcap_{i=1}^{n} (- \circ g \circ f_i)^{-1}[X] &\subseteq (- \circ g)^{-1}[Y] \cap \bigcap_{i=1}^{n} (- \circ g \circ f_i)^{-1}[X] \\ &= (- \circ g)^{-1}\left[Y \cap \bigcap_{i=1}^{n} (- \circ f_i)^{-1}[X]\right] \\ &= \emptyset, \end{aligned}$$

and similarly

$$Z \cap \bigcap_{i=1}^{n} (- \circ g' \circ f'_j)^{-1}[X] = \emptyset.$$

Let $C \in \mathcal{K}$ be such that $C \to (B^*)^A_2$. By Lemma 4.7, Z is thick, so we can find some embedding $h \colon C \hookrightarrow I$ such that for all embeddings $h' \colon B \hookrightarrow C$, $h \circ h' \in Z$. Using h, we define a coloring $\gamma \colon \mathrm{Emb}(A, C) \to \{0, 1\}$ by

$$\gamma(e) = \begin{cases} 1 & \text{if } h \circ e \in X \\ 0 & \text{if } h \circ e \in X'. \end{cases}$$

Since $C \to (B^*)^A_2$, there is some embedding $h^*\colon B^* \hookrightarrow C$ such that set of embeddings $A \to C$ which factor through h^* are monochromatic.

Suppose these embeddings are monochromatic with color 1. This means in particular that for all $1 \leq i \leq n$, we have $\gamma(h^* \circ g \circ f_i) = 1$, i.e., $h \circ h^* \circ g \circ f_i \in X$, so $(h \circ h^*) \in (- \circ g \circ f_i)^{-1}[X]$. But also $h \circ h^* \in Z$, contradicting the fact that $Z \cap \bigcap_{i=1}^n (- \circ g \circ f_i)^{-1}[X] = \emptyset$. The argument in the other case is similar. □

Second proof of Theorem 4.1. We show that any proper age filter family on I extends to an age ultrafilter family. The existence of an age ultrafilter family on I then follows from the fact that $\mathcal{K} = \mathrm{Age}(I)$ has AP (Lemma 3.2), which implies the existence of some proper age filter family on I (Corollary 4.5).

Fix a proper age filter family $\mathcal{F}$ on I, and let $\mathbb{P}$ be the poset of proper age filter families on I extending $\mathcal{F}$, ordered by extension. Then $\mathcal{F} \in \mathbb{P}$, so $\mathbb{P}$ is non-empty.

Let $\{\mathcal{F}_i \mid i \in I\}$ be a non-empty chain in $\mathbb{P}$ with $\mathcal{F}_i = ((F_i)_A)_{A\in\mathcal{K}}$. We define an upper bound $\mathcal{F}'$ by $F'_A = \bigcup_{i\in I}(F_i)_A$ for all $i \in I$. Since, for all $A \in \mathcal{K}$, the set of proper filters on $\mathrm{Emb}(A, I)$ containing F_A is closed under unions of chains, it remains to check the pushforward property. Let $f\colon A \hookrightarrow B$ be an embedding, and let $X \subseteq \mathrm{Emb}(A, I)$. Then $X \in F'_A$ if and only if $X \in (F_i)_A$ for some $i \in I$. But this happens if and only if $(- \circ f)^{-1}[X] \in (F_i)_B$ for some $i \in I$, if and only if $(- \circ f)^{-1}[X] \in F'_B$. Thus $\mathcal{F}' \in \mathbb{P}$.

By Zorn's Lemma, $\mathbb{P}$ has a maximal element $\mathcal{U} = (U_A)_{A\in\mathcal{K}}$. Since $\mathcal{U}$ is a proper age filter family on I extending $\mathcal{F}$, it remains to show that U_A is an ultrafilter for all $A \in \mathcal{K}$. So let $X \subseteq \mathrm{Emb}(A, I)$ be any set. By Lemma 4.8, we can extend $\mathcal{U}$ to a proper age filter family $\mathcal{F}'$ on I which contains either X or its complement. By maximality, $\mathcal{F} = \mathcal{U}$, so U_A contains X or its complement. Thus U_A is an ultrafilter, and $\mathcal{U}$ is an age ultrafilter family on I. □

Corollary 4.9. *Suppose $\mathcal{K} = \mathrm{Age}(I)$ is a Ramsey class. Then any family $\mathcal{G} = (G_A)_{A\in\mathcal{K}}$ with PFIP extends to an age ultrafilter family.*

Proof. By Corollary 4.5, $\mathcal{G}$ generates a proper age filter family, and we showed in the second proof of Theorem 4.1 that any proper age filter family extends to an age ultrafilter family. □

Corollary 4.10. *Suppose $\mathcal{K} = \mathrm{Age}(I)$ is a Ramsey class, and let $\mathcal{G} = (G_A)_{A\in\mathcal{K}}$ be a family with PFIP. Let $\mathcal{I} = (a_i)_{i\in I}$ be a family of tuples from $\mathbb{M}$, and let $D \subseteq \mathbb{M}$ be a small set. Then for any L'-structure J with $\mathrm{Age}(J) \subseteq \mathcal{K}$, there is a family $\mathcal{J}$ of J-indexed indiscernibles over D such that $\mathcal{J}$ realizes $\mathrm{EM}_{\mathcal{G}}(\mathcal{I}/D)$.*

Proof. By Corollary 4.9, $\mathcal{G}$ extends to an age ultrafilter family $\mathcal{U}$. By Lemma 2.2, $\mathrm{EM}_{\mathcal{U}}(\mathcal{I}/D)$ is a complete $\mathcal{K}$-type over D extending $\mathrm{EM}_{\mathcal{G}}(\mathcal{I}/D)$. So any J-indexed realization of this $\mathcal{K}$-type is a family of J-indexed indiscernibles over D realizing $\mathrm{EM}_{\mathcal{G}}(\mathcal{I}/D)$. □