

# First-order logic and cologic over a category

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**Goal:** Generalize first-order logic, in order to widen the scope of model theory to objects in more general categories (especially categories of profinite objects — “cologic”).

Here’s the plan:

- Classical first-order logic - what are we generalizing?
- The syntax and semantics of  $\text{FO}_{\mathcal{C}}$
- Example:  $\text{FO}_{\mathcal{C}} =$  first-order, when  $\mathcal{C} =$  finitely presentable  $\mathcal{L}$ -structures
- Locally finitely presentable categories
- Example: The “cologic” of profinite groups
- Hyperdoctrines, proofs, and completeness
- Adding relational structure
- Example: Homogeneous “costructures”

# Classical first-order logic: Languages

First-order logic starts with a *language*  $\mathcal{L}$ , consisting of a set of function symbols  $\mathcal{F}$  and a set of relation symbols  $\mathcal{R}$ .

Similarly, each function symbol  $f \in \mathcal{F}$  has an *arity*  $\text{ar}(f) \in \mathbb{N}$ .  
0-ary function symbols are called *constant symbols*.

Each relation symbol  $R \in \mathcal{R}$  has an arity  $\text{ar}(R) \in \mathbb{N}$ .  
0-ary relation symbols are called *proposition symbols*.

Example: The language of ordered groups is  $\mathcal{L}_{\text{OG}} = \{e, \cdot, {}^{-1}, \leq\}$ .

- $e$  is a constant symbol,
- $\cdot$  is a binary function symbol,
- ${}^{-1}$  is a unary function symbol, and
- $\leq$  is a binary relation symbol.

# Classical first-order logic: Terms

Let  $\bar{x} = (x_1, \dots, x_n)$  be a **finite** tuple of variables (from some infinite set of variables in the background). We call this a *context*.

Given a language  $\mathcal{L}$  and a context  $\bar{x}$ , the set of  $\mathcal{L}$ -terms in context  $\bar{x}$  is defined inductively:

- A variable  $x_i$  is a term.
- A constant symbol  $c$  is a term.
- If  $f$  is a function symbol and  $t_1, \dots, t_{\text{ar}(f)}$  are terms, then  $f(t_1, \dots, t_{\text{ar}(f)})$  is a term.

Example:  $\mathcal{L}_{\text{OG}}$ -terms in context  $(x, y, z)$  include:

- $e$
- $x \cdot (y \cdot z)$
- $x^{-1} \cdot e^{-1}$ , etc.

# Classical first-order logic: Formulas

An *atomic formula* in context  $\bar{x}$  is:

- $\top$  or  $\perp$ ,
- $t_1 = t_2$ , where  $t_1$  and  $t_2$  are formulas in context  $\bar{x}$ , or
- $R(t_1, \dots, t_{\text{ar}(R)})$ , where  $R$  is a relation symbol and  $t_1, \dots, t_{\text{ar}(R)}$  are terms in context  $\bar{x}$ .

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A *formula* in context  $\bar{x}$  is:

- $(\varphi \wedge \psi)$ , where  $\varphi$  and  $\psi$  are formulas in context  $\bar{x}$ ,
- $(\varphi \vee \psi)$ , where  $\varphi$  and  $\psi$  are formulas in context  $\bar{x}$ ,
- $\neg\varphi$ , where  $\varphi$  is a formula in context  $\bar{x}$ , or
- $\exists y \varphi$ , where  $\varphi$  is a formula in context  $\bar{x}y$ .

Note that the quantifier changes the context!

We also write  $(\psi \rightarrow \theta)$  as shorthand for  $(\neg\psi \vee \theta)$   
and  $\forall y \varphi$  as shorthand for  $\neg\exists y \neg\varphi$ .

# Classical first-order logic: Semantics

Given a language  $\mathcal{L}$ , an  $\mathcal{L}$ -structure is a set  $A$  (the domain) together with interpretations of the symbols in  $\mathcal{L}$ :

- A function  $f^A: A^{\text{ar}(f)} \rightarrow A$  for every function symbol  $f$ .
- A relation  $R^A \subseteq A^{\text{ar}(R)}$  for every relation symbol  $R$ .

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If  $A$  is an  $\mathcal{L}$ -structure and  $\bar{x}$  is a context, an *interpretation* of  $\bar{x}$  is a function  $\bar{x} \rightarrow A$ , i.e. an assignment of elements of  $A$  to the variables in  $\bar{x}$ . We write  $A^{\bar{x}}$  for the set of interpretations of  $\bar{x}$  in  $A$ .



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$\mathcal{L}$ -terms and formulas have an obvious semantics in  $\mathcal{L}$ -structures:

- Every  $\mathcal{L}$ -term  $t$  in context  $\bar{x}$  gives rise to a function  $t^A: A^{\bar{x}} \rightarrow A$
- Every  $\mathcal{L}$ -formula  $\varphi$  in context  $\bar{x}$  defines a subset  $\varphi(A) \subseteq A^{\bar{x}}$ .

If an interpretation  $\bar{a}$  is in  $\varphi(A)$ , we write  $A \models \varphi(\bar{a})$ .

## Example: Semantics in ordered groups

Let  $(G, e, \cdot, ^{-1}, \leq)$  be a (right)-ordered group, viewed as an  $\mathcal{L}_{\text{OG}}$ -structure in the obvious way.

- If  $t$  is the term  $((x \cdot y) \cdot x^{-1}) \cdot y^{-1}$ , then  $t^G: G^{(x,y)} \cong G^2 \rightarrow G$  is the commutator function.
- If  $\varphi$  is the formula  $\forall y (x \cdot y = y \cdot x)$ , then  $\varphi(G) \subseteq G^x \cong G$  is the center, i.e. for  $a \in G$ ,  $G \models \varphi(a)$  if and only if  $a \in Z(G)$ .

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A *sentence* is a formula in the empty context. Since there are no variables to interpret, a sentence is either true or false in an  $\mathcal{L}$ -structure.

- $G \models \forall x \forall y \forall z ((x \leq y) \rightarrow (x \cdot z \leq y \cdot z))$ .
- $G \models \forall x \forall y (x \cdot y = y \cdot x)$  if and only if  $G$  is abelian.

# Generalizing first-order logic

The category of finite sets (technically, finite subsets of the infinite set of variables), and its relationship to the category of all sets (domains of structures) is baked into first-order logic.

First-order formulas explore how infinite structures are built up from finite pieces, via interpretations: maps from contexts to domains.

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First-order formulas explore how infinite structures are built up from finite pieces, via interpretations: maps from contexts to domains.

We seek to generalize classical first-order logic by replacing the category of finite sets with a (somewhat) arbitrary category  $\mathcal{C}$  of “contexts”, sitting inside a larger category  $\mathcal{D}$  of “domains”.

Let  $\mathcal{C}$  be a category, called the category of *contexts*.

Define the logic  $\text{FO}_{\mathcal{C}}$  inductively. For every object  $x \in \mathcal{C}$ , a formula in context  $x$  is:

- $\top_x$  or  $\perp_x$ .
- $(\psi \wedge \theta)$ ,  $(\psi \vee \theta)$ , or  $\neg\psi$ , where  $\psi$  and  $\theta$  are formulas in context  $x$ .
- $\exists_f\psi$ , where  $f: x \rightarrow y$  is an arrow in  $\mathcal{C}$  and  $\psi$  is a formula in context  $y$ .

We can also define  $(\psi \rightarrow \theta)$  as  $(\neg\psi \vee \theta)$  and  $\forall_f\psi$  as  $\neg\exists_f\neg\psi$ .

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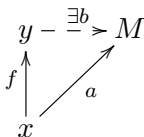
Now suppose  $\mathcal{C}$  is a subcategory of  $\mathcal{D}$ , called the category of *domains*.

If  $x$  is a context in  $\mathcal{C}$  and  $M$  is a domain in  $\mathcal{D}$ , an arrow  $a: x \rightarrow M$  is called an *interpretation* of  $x$  in  $M$ .

# FO<sub>C</sub>: Semantics

We give a semantics in  $\mathcal{D}$  for the logic FO<sub>C</sub> by defining the relation  $M \models \varphi(a)$  inductively. For every domain  $M \in \mathcal{D}$ , every formula  $\varphi$  in context  $x$ , and every interpretation  $a: x \rightarrow M$ ,

- If  $\varphi$  is  $\top_x$ , then  $M \models \varphi(a)$ . If  $\varphi$  is  $\perp_x$ , then  $M \not\models \varphi(a)$ .
- If  $\varphi$  is  $(\psi \wedge \theta)$ , then  $M \models \varphi(a)$  iff  $M \models \psi(a)$  and  $M \models \theta(a)$ .
- If  $\varphi$  is  $(\psi \vee \theta)$ , then  $M \models \varphi(a)$  iff  $M \models \psi(a)$  or  $M \models \theta(a)$ .
- If  $\varphi$  is  $\neg\psi$ , then  $M \models \varphi(a)$  iff  $M \not\models \psi(a)$ .
- If  $\varphi$  is  $\exists_f\psi$ , for  $f: x \rightarrow y$ , then  $M \models \varphi(a)$  iff there exists  $b: y \rightarrow M$  such that  $bf = a$  and  $M \models \psi(b)$ .





## Example: $\mathcal{L}$ -structures

Note that the basic logic  $\text{FO}_{\mathcal{C}}$  has:

- No notion of language (relation and function symbols).
- No interesting atomic formulas!

Nonetheless, if  $\mathcal{C}$  is reasonably complicated,  $\text{FO}_{\mathcal{C}}$  can be interesting.

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Fix a first-order language  $\mathcal{L}$ .

$\mathcal{D}$ , the category of  $\mathcal{L}$ -structures (and  $\mathcal{L}$ -homomorphisms).

$\mathcal{C}$ , the full category of finitely presentable  $\mathcal{L}$ -structures.

### Theorem

*$\text{FO}_{\mathcal{C}}$ , with semantics in  $\mathcal{D}$ , has essentially the same expressive power as first-order logic in the language  $\mathcal{L}$ .*

An  $\mathcal{L}$ -homomorphism is a function  $\sigma: A \rightarrow B$  such that

- For all  $k$ -ary function symbols  $f \in \mathcal{F}$  and all  $a_1, \dots, a_k \in A$ ,  
 $\sigma(f^A(a_1, \dots, a_k)) = f^B(\sigma(a_1), \dots, \sigma(a_k))$ .
- For all  $k$ -ary relation symbols  $R \in \mathcal{R}$  and all  $a_1, \dots, a_k \in A$ ,  
 $(a_1, \dots, a_k) \in R^A \implies (\sigma(a_1), \dots, \sigma(a_k)) \in R^B$ .

# Free $\mathcal{L}$ -structures

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For any finite context  $\bar{x}$ , there is a *free*  $\mathcal{L}$ -structure  $F(\bar{x})$  generated by  $\bar{x}$ .

Domain: terms in context  $\bar{x}$ .

Function symbols: interpreted as term formation

Relation symbols: interpreted as  $\emptyset$ .

Universal property: An  $\mathcal{L}$ -homomorphism  $F(\bar{x}) \rightarrow A$  is determined uniquely by an interpretation  $\bar{a}: \bar{x} \rightarrow A$ , by  $t \mapsto t^A(\bar{a})$ .

# Finitely presentable $\mathcal{L}$ -structures

For any finite context  $\bar{x}$  and finite set  $\Delta$  of atomic formulas in context  $\bar{x}$ , there is a *finitely presented*  $\mathcal{L}$ -structure  $\langle \bar{x} | \Delta \rangle$ .

Universal property: An  $\mathcal{L}$ -homomorphism  $\langle \bar{x} | \Delta \rangle$  is determined uniquely by an interpretation  $\bar{a}: \bar{x} \rightarrow A$  such that  $A \models \delta(\bar{a})$  for all  $\delta \in \Delta$ .

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$\langle \bar{x} | \Delta \rangle$  is obtained as the quotient of  $F(\bar{x})$  by the least congruence (equivalence relation respecting the function symbols) generated by the instances of equality in  $\Delta$ , with relations holding as prescribed by  $\Delta$ .

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An  $\mathcal{L}$ -structure is *finitely presentable* if it is isomorphic to  $\langle \bar{x} | \Delta \rangle$  for some finite  $\bar{x}$  and  $\Delta$ .

# Translation: First-order to $\text{FO}_C$

*Proof of Theorem:*

We translate a first-order formula  $\varphi$  in context  $\bar{x}$  to an  $\text{FO}_C$  formula in context  $F(\bar{x})$ .

- If  $\varphi$  is atomic, let  $\hat{\varphi}$  be  $\exists_q \top_{\langle \bar{x} | \{\varphi\} \rangle}$ , where  $q: F(\bar{x}) \rightarrow \langle \bar{x} | \{\varphi\} \rangle$  is the obvious map.
- If  $\varphi$  is  $\psi \wedge \theta$ ,  $\psi \vee \theta$ , or  $\neg\psi$ , let  $\hat{\varphi}$  be  $\hat{\psi} \wedge \hat{\theta}$ ,  $\hat{\psi} \vee \hat{\theta}$ , or  $\neg\hat{\psi}$ , respectively.
- If  $\varphi$  is  $\exists y \psi$ , where  $\psi$  is a formula in context  $\bar{x}y$ , let  $\hat{\varphi}$  be  $\exists_i \hat{\psi}$ , where  $i: F(\bar{x}) \rightarrow F(\bar{x}y)$  is the obvious map.

$$\begin{array}{ccc} \langle \bar{x} | \{\varphi\} \rangle & \xrightarrow{\exists_q} & M \\ q \uparrow & \nearrow a & \\ T(\bar{x}) & & \end{array}$$

$$\begin{array}{ccc} T(\bar{x}y) & \xrightarrow{\exists_i} & M \\ i \uparrow & \nearrow a & \\ T(\bar{x}) & & \end{array}$$



# Translation: $\text{FO}_C$ to first-order

We translate an  $\text{FO}_C$  formula in context  $A \cong \langle \bar{x} \mid \Delta \rangle$  to a first-order formula in context  $\bar{x}$ .

- If  $\varphi$  is  $\top_A$ , let  $\tilde{\varphi}$  be  $\top$ . If  $\varphi$  is  $\perp_A$ , let  $\tilde{\varphi}$  be  $\perp$ .
- If  $\varphi$  is  $\psi \wedge \theta$ ,  $\psi \vee \theta$ , or  $\neg\psi$ , let  $\tilde{\varphi}$  be  $\tilde{\psi} \wedge \tilde{\theta}$ ,  $\tilde{\psi} \vee \tilde{\theta}$ , or  $\neg\tilde{\psi}$ , respectively.
- If  $\varphi$  is  $\exists_f \psi$ , where  $f: A \rightarrow B$  and  $\psi$  is a formula in context  $B$ :
  - Pick a finite presentation  $B \cong \langle \bar{y} \mid \Delta' \rangle$ .
  - For each  $x_i$ , pick a term  $t_i$  such that  $t_i^B(\bar{y}) = f(x_i)$ .
  - Let  $\tilde{\varphi}$  be

$$\exists y_1 \dots \exists y_n \left( \left( \bigwedge_{\delta \in \Delta'} \delta_i(\bar{y}) \right) \wedge \left( \bigwedge_{i=1}^m x_i = t_i(\bar{y}) \right) \wedge \tilde{\psi}(\bar{y}) \right).$$

$$\begin{array}{ccc} B & \xrightarrow{\exists b} & M \\ f \uparrow & \nearrow a & \\ A & & \end{array}$$

# Locally finitely presentable categories

Question: Returning to general  $\mathcal{C}$  and  $\mathcal{D}$ , what properties do we need to get a well-behaved logic  $\text{FO}_{\mathcal{C}}$ ?

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For the rest of this talk, let's assume:

- $\mathcal{C}$  has finite colimits.
- The objects of  $\mathcal{D}$  are all directed colimits along diagrams in  $\mathcal{C}$ .
- Every object  $x \in \mathcal{C}$  is *finitely presentable* in the categorical sense:  $\text{Hom}_{\mathcal{D}}(x, -)$  preserves directed colimits (every map  $x \rightarrow \varinjlim y_i$  factors through some  $y_i$ ).

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In other words,

- $\mathcal{D}$  is equivalent to  $\text{ind-}\mathcal{C}$ , the free co-completion of  $\mathcal{C}$  under formal directed colimits.
- $\mathcal{D}$  is a *locally finitely presentable category*, and  $\mathcal{C}$  is equivalent to its full subcategory of finitely presentable objects.

## Definition (Gabriel & Ulmer)

A category  $\mathcal{D}$  is *locally finitely presentable* (LFP) if:

- It is co-complete.
- Every object is a directed colimit of finitely presentable objects.
- The full subcategory  $\mathcal{F}$  of finitely presentable objects is essentially small, i.e. there is a set of isomorphism representatives of  $\mathcal{F}$ .

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Examples:

- $\text{Set}$ ;  $\text{Set}^X$ , for any set  $X$ ;  $\mathcal{D}^{\mathcal{B}}$ , for any LFP  $\mathcal{D}$  and small category  $\mathcal{B}$ .
- $\text{Str}_{\mathcal{L}}$ ;  $\text{Grp}$ ;  $\text{Ring}$ ;  $\text{Poset}$ ;  $\text{Cat}$ ;  $\text{Mod}_T$ , where  $T$  is a first-order universal Horn theory.
- The **duals** of  $\text{ProFinSet} \cong \text{Stone} \cong \text{Bool}^{\text{op}}$ ,  $\text{ProFinGrp}$ , and  $\text{pro-}\mathcal{C}$ , the free completion of a small category  $\mathcal{C}$  under formal codirected limits (a.k.a inverse limits).

Many of the standard examples of LFP categories are categories of  $\mathcal{L}$ -structures for appropriate languages  $\mathcal{L}$ .

Similarly to how  $\text{FO}_{\mathcal{C}}$  is essentially equivalent to first-order logic when  $\mathcal{D}$  is the category of  $\mathcal{L}$ -structures, we don't really get anything new when we look at  $\text{FO}_{\mathcal{C}}$  over these categories.

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If  $\mathcal{D}^{\text{op}}$  is LFP, we can form the logic  $\text{FO}_{\mathcal{C}^{\text{op}}}$  with semantics in  $\mathcal{D}^{\text{op}}$ .

In  $\mathcal{D}$ , an interpretation of the context  $x$  in the domain  $M$  is a map  $a: M \rightarrow x$ .

“Coformulas” (formulas of  $\text{FO}_{\mathcal{C}^{\text{op}}}$ ) explore how the domain  $M \in \mathcal{D}$  is built as an inverse limit, via its maps to contexts in  $\mathcal{C}$ .



## Example: The cologic of profinite groups

Consider the concrete example of the LFP category  $\mathcal{D} = \text{ProFinGrp}^{\text{op}}$ , the finitely presentable objects of which are exactly  $\mathcal{C} = \text{FinGrp}^{\text{op}}$ . Here are some examples of properties that can be expressed in  $\text{FO}_{\text{FinGrp}^{\text{op}}}$ :

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Given a finite group  $H$  and a proper subgroup  $H' \subsetneq H$ , let  $i_{H'}$  be the inclusion map. Then the following formula  $\psi_H$  in context  $H$  expresses that a map  $\mathcal{G} \rightarrow H$  is surjective:

$$\bigwedge_{H' \subsetneq H} \neg \exists i_{H'} \top_{H'}$$

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A *sentence* is a formula in the initial context (the terminal group  $0$ ). Letting  $q: C_2 \rightarrow 0$  be the trivial map and  $q': C_4 \rightarrow C_2$  be the quotient map, the following sentence asserts that every quotient of  $\mathcal{G}$  isomorphic to  $C_2$  factors through a quotient of  $\mathcal{G}$  isomorphic to  $C_4$ :

$$\forall q(\psi_{C_2} \rightarrow \exists q' \psi_{C_4})$$

## Example: The cologic of profinite groups

In an influential unpublished paper, “The elementary theory of regularly closed fields”, Cherlin, van den Dries, and Macintyre introduced a “cologic” of profinite groups (e.g. Galois groups) in order to study the model theory of pseudo-algebraically closed (PAC) fields.

## Example: The cologic of profinite groups

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CDM cologic is just ordinary first-order logic on a multi-sorted structure encoding the full inverse system of finite quotients of a profinite group  $\mathcal{G}$ :

- One sort for each  $n \geq 1$ . Sort  $n$  consists of the disjoint union of all finite quotients of  $\mathcal{G}$  of size  $n$ .
- A ternary relation  $\cdot_n$  for each sort  $n$ , such that  $\cdot_n(x, y, z)$  iff all three elements live in the same finite quotient of size  $n$ , and  $x \cdot y = z$ .
- A binary relation  $\pi_{m,n}$  for each pair of sorts  $m \geq n$ , such that  $\pi_{m,n}(x, y)$  iff  $x \in H_1$  of size  $m$ ,  $y \in H_2$  of size  $n$ , and the quotient map  $\pi_{H_2}: \mathcal{G} \rightarrow H_2$  factors through the quotient map  $\pi_{H_1}: \mathcal{G} \rightarrow H_1$ , as  $\pi_{H_2} = \rho \circ \pi_{H_1}$ , and  $\rho(x) = y$ .

## Example: The cologic of profinite groups

We say that two structures are *elementarily equivalent* if they satisfy the same sentences.

### Theorem

*Two profinite groups  $\mathcal{G}$  and  $\mathcal{G}'$  are elementarily equivalent in the sense of CDM cologic if and only if they are elementarily equivalent in the sense of  $\text{FO}_{\text{FinGrp}^{\text{op}}}$ .*

There is not a straightforward translation between formulas of CDM cologic and formulas of  $\text{FO}_{\text{FinGrp}^{\text{op}}}$ . Instead, it is easiest to prove this theorem by an Ehrenfeucht–Fraïssé game argument.

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There is not a straightforward translation between formulas of CDM cologic and formulas of  $\text{FO}_{\text{FinGrp}^{\text{op}}}$ . Instead, it is easiest to prove this theorem by an Ehrenfeucht–Fraïssé game argument.

So the logic  $\text{FO}_{\text{FinGrp}^{\text{op}}}$  provides a natural replacement for the somewhat ad hoc CDM cologic, and it easily generalizes to other kinds of profinite structures.

I'll describe a *sequent calculus* proof system for  $\text{FO}_{\mathcal{C}}$ .

A *sequent* has the form  $\varphi \Rightarrow_x \psi$ , where  $\varphi$  and  $\psi$  are formulas in context  $x$ .

A *proof rule* has the form:

$$\frac{\varphi_1 \Rightarrow_{x_1} \psi_1 \quad \dots \quad \varphi_k \Rightarrow_{x_k} \psi_k}{\varphi_* \Rightarrow_{x_*} \psi_*} \text{ RULE}$$

It means that given the  $k$  sequents above the line (possibly  $k = 0$ ), you can conclude the sequent below the line.



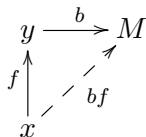
# Substitution

It will be convenient to introduce a new formula-building operation: “substitution”.

Syntax: If  $\varphi$  is a formula in context  $x$  and  $f: x \rightarrow y$  is an arrow in  $\mathcal{C}$ , then  $[\varphi]_f$  is a formula in context  $y$ .

Semantics: Given a domain  $M$  and an interpretation  $b: y \rightarrow M$ ,

$$M \models [\varphi]_f(b) \text{ iff } M \models \varphi(bf).$$



It will follow from our proof rules that every formula is equivalent to one built without any instances of substitution, which is why it was omitted from the original definition.

# Propositional rules

$$\frac{}{\varphi \Rightarrow_x \varphi} \text{ REF} \qquad \frac{\varphi \Rightarrow_x \psi \quad \psi \Rightarrow_x \theta}{\varphi \Rightarrow_x \theta} \text{ TRANS} \qquad \frac{}{\varphi \Rightarrow_x \top_x} \text{ TRUE}$$
$$\frac{\varphi \Rightarrow_x \psi \quad \varphi \Rightarrow_x \theta}{\varphi \Rightarrow_x \psi \wedge \theta} \text{ AND} \qquad \frac{}{\psi \wedge \theta \Rightarrow_x \psi} \text{ AND}_L \qquad \frac{}{\psi \wedge \theta \Rightarrow_x \theta} \text{ AND}_R$$
$$\frac{\psi \Rightarrow_x \varphi \quad \theta \Rightarrow_x \varphi}{\psi \vee \theta \Rightarrow_x \varphi} \text{ OR} \qquad \frac{}{\psi \Rightarrow_x \psi \vee \theta} \text{ OR}_L \qquad \frac{}{\theta \Rightarrow_x \psi \vee \theta} \text{ OR}_R$$
$$\frac{}{\varphi \wedge (\psi \vee \theta) \Rightarrow_x (\varphi \wedge \psi) \vee (\varphi \wedge \theta)} \text{ DIST} \qquad \frac{}{\perp_x \Rightarrow_x \varphi} \text{ FALSE}$$
$$\frac{}{\top_x \Rightarrow_x \varphi \vee \neg \varphi} \text{ NOT}_1 \qquad \frac{}{\varphi \wedge \neg \varphi \Rightarrow_x \perp_x} \text{ NOT}_2$$

# Substitution rules

For all arrows  $f: x \rightarrow y$  and  $g: y \rightarrow z$  in  $\mathcal{C}$ ,

$$\frac{}{\varphi \Leftrightarrow_x [\varphi]_{\text{id}_x}} \text{ID} \quad \frac{}{[\varphi]_{gf} \Leftrightarrow_z [[\varphi]_f]_g} \text{COMP} \quad \frac{\varphi \Rightarrow_x \psi}{[\varphi]_f \Rightarrow_y [\psi]_f} \text{MON}$$

$$\frac{}{\top_y \Rightarrow_y [\top_x]_f} \text{HOM}_{\top} \quad \frac{}{[\perp_x]_f \Rightarrow_y \perp_y} \text{HOM}_{\perp}$$

$$\frac{}{[\psi]_f \wedge [\theta]_f \Rightarrow_y [\psi \wedge \theta]_f} \text{HOM}_{\wedge} \quad \frac{}{[\psi \vee \theta]_f \Rightarrow_y [\psi]_f \vee [\theta]_f} \text{HOM}_{\vee}$$

# Quantifier rules

For every arrow  $f: x \rightarrow y$  in  $\mathcal{C}$ ,

$$\frac{\varphi \Rightarrow_y \psi}{\exists_f \varphi \Rightarrow_x \exists_f \psi} \text{ MON}\exists$$

$$\frac{}{\varphi \Rightarrow_y [\exists_f \varphi]_f} \text{ UNIT}$$

$$\frac{}{\exists_f [\theta]_f \Rightarrow_x \theta} \text{ COUNIT}$$

For every pushout square,

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ g \downarrow & & \downarrow g' \\ z & \xrightarrow{f'} & w \end{array}$$

$$\frac{}{[\exists_f \varphi]_g \Rightarrow_z \exists_{f'} [\varphi]_{g'}} \text{ BC}$$

A *proof* of is a tree of deductions, such that each step has the form of one of our proof rules. The leaves of the tree are the hypothesis of the proof, and the root is the conclusion.

Here is a simple proof of the sequent  $\varphi \Rightarrow_x \varphi \wedge \theta$  from the set of sequents  $\{\varphi \Rightarrow_x \psi, \psi \Rightarrow_x \theta\}$ :

$$\frac{\frac{\varphi \Rightarrow_x \varphi}{\text{REF}} \quad \frac{\frac{\varphi \Rightarrow_x \psi \quad \psi \Rightarrow_x \theta}{\varphi \Rightarrow_x \theta} \text{TRANS}}{\varphi \Rightarrow_x \varphi \wedge \theta} \text{AND}}$$

If there is a proof of  $\sigma$  with hypotheses from the set of sequents  $T$ , we write  $T \vdash \sigma$ .

# Why these proof rules?

## Definition

Let  $\mathcal{B}$  be a category with finite limits. A *first-order (Boolean) hyperdoctrine* over  $\mathcal{B}$  is a functor  $P: \mathcal{B}^{\text{op}} \rightarrow \text{Bool}$ , such that for every arrow  $f: y \rightarrow x$  in  $\mathcal{B}$ , the Boolean homomorphism  $Pf: Px \rightarrow Py$  has a left adjoint, i.e. a monotone map  $\exists_f: Py \rightarrow Px$  such that

$$\varphi \leq_{Py} Pf(\psi) \iff \exists_f \varphi \leq_{Px} \psi,$$

satisfying the Beck-Chevalley condition: For every pullback square in  $\mathcal{B}$ ,

$$\begin{array}{ccc} w & \xrightarrow{f'} & z \\ g' \downarrow & & \downarrow g \\ y & \xrightarrow{f} & x \end{array}$$

and every  $\varphi \in Py$ , we have  $Pg(\exists_f(\varphi)) = \exists_{f'}(Pg'(\varphi))$ .

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Formulas  $\varphi$  and  $\psi$  in context  $x$  are *provably equivalent* if  $\vdash \varphi \Leftrightarrow_x \psi$ .

Our proof rules ensure that for every context  $x$ , the set of formulas in context  $x$  modulo provable equivalence form a Boolean algebra, and this family of Boolean algebras coheres to a first-order hyperdoctrine over  $\mathcal{C}^{\text{op}}$ .

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In fact, since the formulas are generated by the formula-building operations from the atomics  $\top_x$  and  $\perp_x$  (the constants of the Boolean algebras in each context), the logic  $\text{FO}_{\mathcal{C}}$  is the *initial hyperdoctrine* over  $\mathcal{C}^{\text{op}}$ !



# Soundness and completeness

A domain  $M$  *satisfies*  $\varphi \Rightarrow_x \psi$  (written  $M \models \varphi \Rightarrow_x \psi$ ) if and only if, for every interpretation  $a: x \rightarrow M$ , if  $M \models \varphi(a)$ , then  $M \models \psi(a)$ .

Let  $T$  be a set of sequents. We say a domain  $M$  is a *model* of  $T$  (written  $M \models T$ ) if  $M \models \sigma$  for all  $\sigma \in T$ .

For any sequent  $\sigma$ , we say that  $\sigma$  is a *semantic consequence* of  $T$  (written  $T \models \sigma$ ) if  $M \models \sigma$  for all  $M \models T$ .

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## Theorem (Soundness and Completeness)

Let  $T$  be a set of sequents and  $\sigma$  a sequent.

- If  $T \vdash \sigma$ , then  $T \models \sigma$  (Soundness).
- If  $T \models \sigma$ , then  $T \vdash \sigma$  (Completeness).

Soundness is easy: just check that the proof rules are valid.

## Completeness: Proof sketch

To prove completeness, suppose that  $T \not\vdash \sigma$ . We want to prove that  $T \not\models \sigma$ , i.e. we want to build a model  $M \models T$  such that  $M \not\models \sigma$ .

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Build  $M$  as a directed colimit from  $\mathcal{C}$ . For each object  $y$  in the diagram, we will choose an ultrafilter  $F_y$  in the Boolean algebra of formulas in context  $y$  which respects the sequents in  $T$ . These will be the formulas which are true in interpretation  $y$ .

If  $\sigma$  is  $\varphi \Rightarrow_x \psi$ , start the construction with the object  $x$ , and ensure that the ultrafilter  $F_x$  contains  $\varphi$  but not  $\psi$ .

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Whenever a formula  $\exists_f \theta$  is in  $F_y$  for some arrow  $f: y \rightarrow z$ , add this arrow to the diagram, and ensure that  $\theta$  is in  $F_z$ , along with  $[\rho]_f$  for all  $\rho \in F_y$ . Close up to a directed diagram by pushouts.

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Let  $M$  be the directed colimit of the resulting diagram and  $a_y: y \rightarrow M$  the inclusion map. Verify that  $M \models \varphi(a_y)$  if and only if  $\varphi \in F_y$ .

## Corollary (Compactness)

*Let  $T$  be a set of sequents. Then  $T$  has a model if and only if every finite subset of  $T$  has a model.*

*Proof:* If  $T$  has a model, then every finite subset of  $T$  has a model. Conversely, if  $T$  has no model, then  $T$  is inconsistent:  $T \models \top \Rightarrow_0 \perp$ . But then  $T \vdash \top \Rightarrow_0 \perp$ , and *proofs are finite*, so already a finite subset of  $T$  is inconsistent, and this finite subset has no model.

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## Corollary (The categorical interpretation)

*The initial hyperdoctrine over  $\mathcal{C}^{\text{op}}$  has a natural semantics in  $\text{ind-}\mathcal{C}$ , for which it is sound and complete.*



# Adding relational structure

In  $\text{FO}_{\mathcal{C}}$ , there are no interesting atomic formulas: all the complexity comes from the category of contexts. Quantifiers are complicated.

On the other hand, in ordinary first-order logic, the category of contexts is very simple (essentially  $\text{FinSet}$ ), and so are the quantifiers. The complexity comes from the added structure: relation and function symbols.

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If we augment  $\text{FO}_{\mathcal{C}}$  by adding atomic formulas, we can sometimes work with a simpler category of contexts (e.g. in the case of classical first-order logic: the category of finite sets instead of the category of finitely presentable  $\mathcal{L}$ -structures).

# Relational languages and structures

Fix categories  $\mathcal{C}$  and  $\mathcal{D}$  as before.

A *relational language*  $\mathcal{L}$  over  $\mathcal{C}$  is a set of relation symbols. Each relation symbol  $R \in \mathcal{L}$  has an *arity*  $\text{ar}(R) \in \mathcal{C}$ .

An  $\mathcal{L}$ -*structure* is a domain  $M$  in  $\mathcal{D}$ , together with, for every  $R \in \mathcal{L}$ , a subset  $R^M \subseteq \text{Hom}_{\mathcal{D}}(\text{ar}(R), M)$ .

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**Syntax:** We add new atomic formulas. If  $R$  is a relation symbol and  $f: \text{ar}(R) \rightarrow x$  is an arrow in  $\mathcal{C}$ , then  $[R]_f$  is a formula in context  $x$ .

**Semantics:** Given an  $\mathcal{L}$ -structure  $M$  and an interpretation  $a: x \rightarrow M$ ,

$$M \models [R]_f(a) \text{ iff } af \in R^M.$$

$$\begin{array}{ccc} x & \xrightarrow{a} & M \\ f \uparrow & \nearrow \text{dashed} & \\ \text{ar}(R) & & \end{array}$$

The diagram illustrates a commutative triangle. At the bottom-left is the object  $\text{ar}(R)$ . A solid vertical arrow labeled  $f$  points up to the object  $x$ . A solid horizontal arrow labeled  $a$  points right from  $x$  to the object  $M$ . A dashed diagonal arrow labeled  $af$  points from  $\text{ar}(R)$  directly to  $M$ .

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The formulas of  $\text{FO}_{\mathcal{C}}(\mathcal{L})$  are generated by the relation symbols in  $\mathcal{L}$ , subject only to the relations imposed by the definition of a hyperdoctrine. So we can describe the logic  $\text{FO}_{\mathcal{C}}(\mathcal{L})$  as the *free hyperdoctrine* over  $\mathcal{C}^{\text{op}}$  generated by  $\mathcal{L}$ .

## Corollary (The categorical interpretation)

*The free hyperdoctrine over  $\mathcal{C}^{\text{op}}$  generated by  $\mathcal{L}$  has a natural semantics in the category of  $\mathcal{L}$ -structures in  $\text{ind-}\mathcal{C}$ , for which it is sound and complete.*

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## Theorem

*Let  $\mathcal{L}$  be a relational language over  $\mathcal{C}$ . The category  $\text{Str}_{\mathcal{L}}$  of  $\mathcal{L}$ -structures is itself a locally finitely presentable category, and  $\text{FO}_{\text{fpStr}_{\mathcal{L}}}$  has essentially the same expressive power as  $\text{FO}_{\mathcal{C}}(\mathcal{L})$ .*

## Example: Stone spaces

A Stone space is a totally disconnected compact Hausdorff space. The category of Stone space is LFP:  $\mathcal{D} = \text{Stone}^{\text{op}} \cong \text{ProFinSet}^{\text{op}}$ , and the finitely presentable objects are  $\mathcal{C} = \text{FinSet}^{\text{op}}$  (view a finite set as a discrete Stone space).



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If  $S$  is a Stone space and  $x$  is a finite set, an interpretation of  $x$  in  $S$  is a continuous map  $S \rightarrow x$ ; equivalently, a partition of  $S$  into  $|x|$  clopen pieces.

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An  $x$ -ary relation picks out a subset of the continuous maps  $S \rightarrow x$ . For example, if  $S$  comes equipped with a measure, there is a 2-ary relation which holds of  $f: S \rightarrow 2$  if and only if the two clopen pieces  $f^{-1}(\{0\})$  and  $f^{-1}(\{1\})$  have equal measure.

# Costructures

Let  $\mathcal{L}$  be a relational language over  $\text{FinSet}^{\text{op}}$ . A *costructure* is an  $\mathcal{L}$ -structure, i.e. a Stone space equipped with interpretations of the “corelations” in  $\mathcal{L}$ .

The natural notion of morphism of costructures is *coembedding*: a surjective map which preserves the interpretations of the corelations.

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The natural notion of morphism of costructures is *coembedding*: a surjective map which preserves the interpretations of the corelations.

A costructure  $M$  is *cohomogeneous* if for any finite costructure  $A$  and any two coembeddings  $f: M \rightarrow A$  and  $f': M \rightarrow A$ , there is an automorphism  $\sigma: M \cong M$  such that  $f = f'\sigma$ .

$$\begin{array}{ccc} M & \overset{\sigma}{\dashrightarrow} & M \\ & \searrow f & \swarrow f' \\ & & A \end{array}$$

# Cohomogeneous costructures

The notion of costructure has appeared in the work of Solecki on dual Ramsey theory and projective Fraïssé theory, and it comes to the fore in the recent work of Panagiotopoulos.

## Theorem (Panagiotopoulos, '16)

- 1 *Let  $S$  be any second-countable Stone space. Then a subgroup  $G$  of  $\text{Homeo}(S)$  is closed in the compact-open topology if and only if there is a language  $\mathcal{L}$  and a cohomogeneous  $\mathcal{L}$ -costructure  $M$  with domain  $S$  such that  $G = \text{Aut}(M)$ .*
- 2 *Let  $Y$  be any second-countable compact Hausdorff space. Then there is a cohomogeneous costructure  $M$  which admits a canonical equivalence relation  $\sim$ , and  $M/\sim$  is homeomorphic to  $Y$ .*

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## Theorem

*A costructure is cohomogeneous if and only if its  $\text{FO}_{\text{FinSet}^{\text{op}}}(\mathcal{L})$ -theory is  $\aleph_0$ -categorical (once this is properly defined) and eliminates quantifiers.*

- 1 Generalize to categories which are not locally finitely presentable. In particular, it would be interesting to generalize from Stone spaces to arbitrary compact Hausdorff spaces, and from profinite groups to compact groups.
- 2 I described how to augment  $\text{FO}_C$  with relational structure. It is also possible to add algebraic structure (i.e. function symbols). This provides a language for the categorical theory of algebras for a functor (and coalgebras for a functor, in the cologic setting). With Larry Moss, we are exploring applications to coalgebras in modal logic.
- 3 In concrete profinite structures, both the tuples (maps  $x \rightarrow M$ ) and cotuples (maps  $M \rightarrow x$ ) are interesting. It would be interesting to develop a logic which talks about both at once.