Notes on the Stability Spectrum

Alex Kruckman

06/01/12

1 Introduction

I wrote this document while preparing for my qualifying exam in model theory. The goal was to clarify for myself some of the many equivalent definitions that appear in stability theory, including where countable language assumptions are necessary, and to collect the arguments that go into Shelah's stability spectrum for countable theories in a compact, readable, and hopefully memorable form.

In Section 2, I have attempted to review all of the definitions and facts about these definitions which are used in the remainder and which are specific to stability theory. This was, again, partially done to test myself on this material before the qualifying exam.

Section 3 contains four theorems, characterizing stable formulas (1), stable theories (2), superstable theories (3), and totally transcendental theories (4). In Theorems 2, 3, and 4, the last equivalent statement in each serves to drive a hard line between the unstable and the stable theories (2.(3)), the strictly stable and the superstable theories (3.(5)), and the strictly superstable and the totally transcendental theories (4.(3)). Together, these theorems give a nice classification of all countable theories into one of these four types, and this result (the stability spectrum for countable theories) is summed up in the Corollary at the end of the document.

But the implications 3.(5) $\Rightarrow$ (2) and 4.(3) $\Rightarrow$ (1) only hold in countable theories (as noted in the statements of the theorems). In uncountable theories, the stability spectrum is significantly more complicated (although a version of 3.(5) $\Rightarrow$ (2) still holds, using a cardinal other than ω). Since the equivalence of (4) in Theorem 3 and (2) in Theorem 4 go through these implications, this leaves open the question of whether 3.(4) is equivalent to superstable and whether 4.(2) is equivalent to superstable.

Probably these answers are available in Shelah and elsewhere, but I haven't thought hard about them or gone looking. If you know or find these answers, or if you see any way to improve the arguments given here, or if you find mistakes, please let me know: `kruckman@gmail.com`.

2 Review of Terminology

Fix a complete theory T. We work in a universe-sized, saturated, strongly homogeneous model of T, $\mathbb{M}$. All elements come from $\mathbb{M}$, all sets are subsets of $\mathbb{M}$, all models are elementary substructures of $\mathbb{M}$, and all ordinals and cardinals are smaller than the cardinality of $\mathbb{M}$. I will realize types and move sets by automorphisms in $\mathbb{M}$ without explicitly mentioning the saturation or strong homogeneity. The notation $\models \phi$ means $\mathbb{M} \models \phi$. When discussing forking, it is convenient to talk about algebraic closure in $\mathbb{M}^{\mathrm{eq}}$, denoted $\mathrm{acl}^{\mathrm{eq}}$.

Definition. Let x and y be tuples of variables, and let $\phi(x, y)$ be a formula.

- A *ϕ-formula* over A is a boolean combination of formulas of the form $\phi(x, a)$ and $\neg\phi(x, a)$ with a from A.
- A *ϕ-type* over A is a maximal consistent set of ϕ-formulas over A.
- $S_x^\phi(A)$ is the set of all ϕ-types over A.
- $S_x(A)$ is the set of all types over A with variables from x.
- ϕ is *stable in* κ if for all A with $|A| \leq \kappa$, $|S_x^\phi(A)| \leq \kappa$.
- T is *stable in* κ if for all A with $|A| \leq \kappa$ and all tuples x, $|S_x(A)| \leq \kappa$.
- St-Spec(T), the *stability spectrum* of T, is the class of all infinite cardinals κ such that T is stable in κ.

Definition. $\phi(x, y)$ has the *order property* if for all $n \in \omega$ there are sequences $a_1, \ldots, a_n$ and $b_1, \ldots b_n$ such that $\models \phi(a_i, b_j)$ if and only if $i \leq j$.

Note that replacing $\leq$ with $<$, $\geq$, or $>$ in the above definition is equivalent.

Definition. A $\phi(x, y)$-type p over A is *definable over* B if there is a formula $\psi(y)$ over B such that for all $a \in A$, $\phi(x, a) \in p$ if and only if $\models \psi(a)$.

A complete type p over A is *definable over* B if every ϕ-type obtained by restricting p to ϕ-formulas is definable over B.

Definition. (T stable in some κ) Let $A \subseteq B$, $p \in S_x(A)$, $q \in S_x(B)$, $p \subseteq q$. We say that q is a *non-forking extension* of p, written $q \sqsupseteq p$, if some extension of q to $\mathrm{acl}^{\mathrm{eq}}(B)$ is definable over $\mathrm{acl}^{\mathrm{eq}}(A)$ (in particular this provides a definition of q over $\mathrm{acl}^{\mathrm{eq}}(A)$). Otherwise q forks over A. We write $c \mathop{\smile\!\!\!\!\!|}_A B$ for $\mathrm{tp}(c/A \cup B) \sqsupseteq \mathrm{tp}(c/A)$ and say c is free from B over A.

We will use the following properties of forking:

- **Extension:** If p is a type over A and $A \subseteq B$, then p has an non-forking extension to a type over B. Equivalently, given a tuple c and a set B, there is a tuple d with $d \equiv_A c$ and $d \mathop{\smile\!\!\!\!\!|}_A B$.
- **Finite character:** If $a \not\mathop{\smile\!\!\!\!\!|}_A B$, then there is a finite tuple b from B such that $a \not\mathop{\smile\!\!\!\!\!|}_A b$.
- **Monotonicity/Transitivity**: If $p \subseteq q \subseteq r$, then $p \sqsubseteq r$ if and only if $p \sqsubseteq q$ and $q \sqsubseteq r$. Equivalently, if $A \subseteq B \subseteq C$, $d \mathop{\smile\!\!\!\!\!|}_A C$ if and only if $d \mathop{\smile\!\!\!\!\!|}_A B$ and $d \mathop{\smile\!\!\!\!\!|}_B C$
- **Symmetry**: If $c \mathop{\smile\!\!\!\!\!|}_A b$, then $b \mathop{\smile\!\!\!\!\!|}_A c$.

The ranks in the following definitions take values in the ordinals together with -1 and ∞, where -1 is less than every ordinal and ∞ is greater than every ordinal. One can give more general definitions of all three (allowing for evaluation on partial types), but we won't need those here.

Definition. Given a formula $\phi(x, y)$, the rank R^ϕ of a formula $\psi(x)$ is defined as follows:
$R^\phi(\psi) \geq 0$ if ψ is consistent.
$R^\phi(\psi) \geq \lambda$ a limit ordinal if $R^\phi(\psi) \geq \alpha$ for all $\alpha < \lambda$.
$R^\phi(\psi) \geq \alpha+1$ if there is a tuple b such that $R^\phi(\psi \wedge \phi(x, b)) \geq \alpha$ and $R^\phi(\psi \wedge \neg\phi(x, b)) \geq \alpha$.

For an ordinal μ, $R^\infty(\psi) \geq n$ is equivalent to the ability to build a ϕ-tree of height μ whose leaves satisfy ψ, as in the following definition:

Definition. Let μ be an ordinal and $\phi(x, y)$ and $\psi(x)$ formulas. A ϕ-tree of height μ whose leaves satisfy ψ is the following configuration: a tuple b_η for every $\eta \in 2^{<\mu}$ (view elements of $2^{<\mu}$ as binary sequences of length less than n), together with a tuple a_σ for every $\sigma \in 2^\mu$, such that for every σ, $\models \psi(a_\sigma)$, and for every initial segment η of σ, $\models \phi(a_\sigma, b_\eta)$ if the next bit of σ after the end of η is a 1 (i.e. a_σ is a right descendant of b_η on the tree), or $\models \neg\phi(a_\sigma, b_\eta)$ if the next bit of σ after the end of η is a 0 (a_σ is a left descendant of b_η).

For μ finite, the existence of such a (finite) configuration is expressible by a single formula, whose parameters are only the parameters appearing in ψ.

Note also that if there are ϕ-trees of arbitrarily large finite height, then by compactness there are ϕ-trees of arbitrary ordinal height. As a consequence (although we will not explicitly use this), $R^\phi(\psi) \geq \omega$ implies $R^\phi(\psi) = \infty$.

Definition. The *Morley rank* RM of a formula $\phi(x)$ is defined as follows:
$\mathrm{RM}(\phi) \geq 0$ if ϕ is consistent.
$\mathrm{RM}(\phi) \geq \lambda$ a limit ordinal if $\mathrm{RM}(\phi) \geq \alpha$ for all $\alpha < \lambda$.
$\mathrm{RM}(\phi) \geq \alpha + 1$ if there are formulas $\langle \psi_i(x) \rangle_{i \in \omega}$ such that $\psi_i \to \neg\psi_j$ for all $i \neq j$ and $\mathrm{RM}(\phi \wedge \psi_i) \geq \alpha$ for all i.

The set $S_x(A)$ for any set A carries the structure of a compact, totally disconnected, Hausdorff space, with topology generated by the basic (clopen) sets $U_\phi = \{p \in S_x(A) \mid \phi \in p\}$.

In if $M \models T$ is ω-saturated, the Morley rank $\mathrm{RM}(\phi)$ agrees with the Cantor-Bendixson rank of the subspace U_ϕ of $S_x(M)$.

For R^ϕ-rank and Morley rank, it is easy to check by induction on α that if $\theta \to \psi$, then $R^\phi(\theta) \leq R^\phi(\psi)$ and $\mathrm{RM}(\theta) \leq \mathrm{RM}(\psi)$.

Definition. The *Lascar rank* U of a complete type p is defined as follows:
$U(p) \geq 0$ always
$U(p) \geq \lambda$ a limit ordinal if $U(p) \geq \alpha$ for all $\alpha < \lambda$.
$U(p) \geq \alpha + 1$ if there is a forking extension $q \supset p$ with $U(q) \geq \alpha$.

If p is a type over A and c realizes p, we can equivalently phrase the successor condition as: there is $B \supset A$ such that $c \not\mathop{\smile\!\!\!\!\!\!|}\,_A B$ and $U(\mathrm{tp}(c/B)) \geq \alpha$ (move a realization of q to c by an automorphism fixing A).

If $U(p) < \infty$ and $p \subseteq q$, then $U(p) = U(q)$ if $p \sqsubseteq q$ and $U(p) > U(q)$ otherwise.

3 Content

Theorem 1 (ϕ is stable)**.** The following are equivalent for a formula $\phi(x, y)$:

(1) ϕ does not have the order property.

(2) ϕ is stable in κ for all κ.

(3) ϕ is stable in κ for some κ.

(4) $R^\phi(x = x) < \omega$ (if x is a tuple of variables, $x = x$ is $x_1 = x_1 \wedge \cdots \wedge x_n = x_n$).

(5) All ϕ-types over a set A are definable over A.

Proof.

$$\begin{tikzcd}
(1) \arrow[d, Rightarrow] & (4) \arrow[l, Rightarrow] \arrow[r, Rightarrow] & (5) \arrow[dl, Rightarrow] \\
(2) \arrow[r, Rightarrow] & (3) \arrow[u, Rightarrow] &
\end{tikzcd}$$

(1) $\Rightarrow$ (2): This is the hardest implication. I'll prove the contrapositive - assume ϕ is unstable in some κ. Let A be a set with $|A| = \kappa < |S_x^\phi(A)|$. Take $C = \{c_i \mid i \in \kappa^+\}$, a set of tuples with distinct ϕ-types over A.

Let $\widetilde{\phi}$ be the same formula $\phi(x, y)$, but with the x variables viewed as the parameters, so a $\widetilde{\phi}$-type has free variables y.

Let p be a ϕ-type over A and B a set. We say p splits over B if there are a and a' from A such that $\mathrm{tp}^{\widetilde{\phi}}(a/B) = \mathrm{tp}^{\widetilde{\phi}}(a'/B)$, but $\phi(x, a) \in p$ and $\neg\phi(x, a') \in p$.

Build a sequence of sets $\langle A_i \rangle_{i \in \omega}$, each of size κ, as follows: Let $A_0 = A$. Given A_n, let A_{n+1} be A_n together with realizations of all ϕ-types and $\widetilde{\phi}$-types over finite subsets $B \subseteq_{\mathrm{fin}} A_n$. Since there are at most κ many such finite subsets and there are finitely many ϕ-types and $\widetilde{\phi}$-types over any finite set, we can construct A_{n+1} of size κ.

Claim: There is a $c \in C$ such that for all $n \in \omega$ and for all finite subsets $B \subseteq_{\mathrm{fin}} A_n$, $\mathrm{tp}^\phi(c/A_{n+1})$ splits over B.

Suppose not. Then for all $c \in C$, there is $n \in \omega$ and $B \subseteq_{\mathrm{fin}} A_n$ such that $\mathrm{tp}^\phi(c/A_{n+1})$ does not split over B.

Now we will refine our set C to make it more homogeneous. The association of an n to each $c \in C$ gives a map $C \to \omega$, so we can pick an n whose preimage in C has size κ^+, and replace C with this subset. Similarly, the association of a $B \subseteq_{\mathrm{fin}} A_n$ to each $c \in C$ gives a map $C \to \mathcal{P}_{\mathrm{fin}}(A_n)$. Since $|\mathcal{P}_{\mathrm{fin}}(A_n)| = \kappa$, we can pick a $B \subseteq_{\mathrm{fin}} A_n$ whose preimage in C has size κ^+, and replace C with this subset.

There are only finitely many $\widetilde{\phi}$-types over B, all of which are realized in A_{n+1}, so we can pick $\widetilde{B} \subseteq_{\mathrm{fin}} A_{n+1}$ containing realizations of all these types. Now there are only finitely many ϕ-types over $\widetilde{B}$, so we can pick distinct $c_1, c_2 \in C$ such that $\mathrm{tp}^\phi(c_1/\widetilde{B}) = \mathrm{tp}^\phi(c_2/\widetilde{B})$. We also know that $\mathrm{tp}^\phi(c_1/A) \neq \mathrm{tp}^\phi(c_2/A)$, since they are distinct elements of C, and that $\mathrm{tp}^\phi(c_1/A_{n+1})$ and $\mathrm{tp}^\phi(c_2/A_{n+1})$ do not split over B. Now we can forget about C and work with c_1 and c_2.

Since $\mathrm{tp}^\phi(c_1/A) \neq \mathrm{tp}^\phi(c_2/A)$, there is a from A with $\models \phi(c_1, a)$ but $\models \neg\phi(c_2, a)$ (switch the numbering of c_1 and c_2 if necessary). Let b from $\widetilde{B}$ realize $\mathrm{tp}^{\widetilde{\phi}}(a/B)$. Now we have a from $A \subseteq A_{n+1}$ and b from $\widetilde{B} \subseteq A_{n+1}$, with $\mathrm{tp}^{\widetilde{\phi}}(a/B) = \mathrm{tp}^{\widetilde{\phi}}(b/B)$. Since $\mathrm{tp}^\phi(c_1/A_{n+1})$ does not split over B, and $\models \phi(c_1, a)$, we must have $\models \phi(c_1, b)$. Similarly, since $\mathrm{tp}^\phi(c_2/A_{n+1})$ does not split over B, and $\models \neg\phi(c_2, a)$, we must have $\models \neg\phi(c_2, b)$. But b is from $\widetilde{B}$, contradicting $\mathrm{tp}^\phi(c_1/\widetilde{B}) = \mathrm{tp}^\phi(c_2/\widetilde{B})$.

Now we'll use the tuple c provided by the claim to construct witnesses to the order property. By induction, define tuples u_n, v_n, w_n from A_{2n+2} for $n \in \omega$, where u_n and v_n can be substituted for y and w_n can be substituted for x in $\phi(x, y)$.

Suppose we have defined u_k, v_k, w_k for $k < n$. Let $B_n = \bigcup_{k<n}\{w_k\} \subseteq_{\text{fin}} A_{2n}$. Since $\mathrm{tp}(c/A_{2n+1})$ splits over B_n, we can find u_n and v_n from A_{2n+1} with $\mathrm{tp}^{\widetilde{\phi}}(u_n/B_n) = \mathrm{tp}^{\widetilde{\phi}}(v_n/B_n)$, but $\models \phi(c, u_n)$ and $\models \neg\phi(c, v_n)$. Now let $B'_n = \cup_{k\leq n}\{u_n, v_n\} \subseteq_{\text{fin}} A_{2n+1}$. Since all ϕ-types over finite subsets of A_{2n+1} are realized in A_{2n+2}, we can find w_n from A_{2n+2} realizing $\mathrm{tp}^\phi(c/B'_n)$.

Now if $i \geq j$, we have $\models \phi(w_i, u_j)$ and $\models \neg\phi(w_i, v_j)$, since $\models \phi(c, u_j)$ and $\models \neg\phi(c, v_j)$, and u_j and v_j are in B'_i. On the other hand, if $i < j$, I claim that either $\models \neg\phi(w_i, u_j)$ or $\models \phi(w_i, v_j)$. Indeed, if $\models \phi(w_i, u_j)$, then since $\mathrm{tp}^{\widetilde{\phi}}(u_j/B_j) = \mathrm{tp}^{\widetilde{\phi}}(v_j/B_j)$, and w_i is in B_j, $\models \phi(w_i, v_j)$.

So we can define a function $[\omega]^2 \to 2$, taking a pair $i < j$ to 0 if $\models \neg\phi(w_i, u_j)$ and 1 otherwise. By Ramsey's Theorem, we can pick an infinite homogeneous subset of ω, which induces an infinite subsequence of $\langle u_n, v_n, w_n\rangle_{i\in\omega}$. Replacing the original sequence with the subsequence, we either have that for all $i < j$, $\models \neg\phi(w_i, u_j)$, in which case $\langle w_n, u_n\rangle_{i\in\omega}$ witnesses the order property (with ϕ corresponding to $\geq$), or for all $i < j$, $\models \phi(w_i, v_j)$, in which case $\langle w_n, v_n\rangle_{i\in\omega}$ witnesses the order property (with ϕ corresponding to $<$).

(2) $\Rightarrow$ (3): There's no work to be done here.

(3) $\Rightarrow$ (4): We'll prove the contrapositive. Suppose $R^\phi(x = x) \geq \omega$. Then we can build ϕ-trees of any finite height, so by compactness, we can build ϕ-trees of any ordinal height.

Let κ be any cardinal. Let μ be the least cardinal such that $2^\mu > \kappa$. Such a μ exists, and in fact $\mu \leq \kappa$, since $2^\kappa > \kappa$. Now build a ϕ-tree of height μ, consisting of $\{b_\eta \mid \eta \in 2^{<\mu}\}$ and $\{a_\sigma \mid \sigma \in 2^\mu\}$. Let $A = \{b_\eta \mid \eta \in 2^{<\mu}\}$. Then $|A| = |2^{<\mu}| \leq \sum_{\alpha<\mu} |2^\alpha| \leq \kappa \cdot \kappa = \kappa$, since $\mu \leq \kappa$ and $|2^\alpha| \leq \kappa$ for all $\alpha < \mu$. But for $\sigma \neq \tau$, a_σ and a_τ have distinct ϕ-types, witnessed by the b_η at the first position in which the sequences σ and τ differ. So $|S^\phi_x(A)| \geq 2^\mu > \kappa$, and ϕ is unstable in κ.

(4) $\Rightarrow$ (1): Again, we'll prove the contrapositive. Suppose ϕ has the order property. Then for any n, we can choose sequences $a_1, \ldots, a_{2^n}$ and $b_1, \ldots, b_{2^n}$ ordered by ϕ, and we can arrange these tuples into a ϕ-tree of height n witnessing $R^\phi(x = x) \geq n$. So we see that $R^\phi(x = x) \geq \omega$.

(4) $\Rightarrow$ (5): Let p be a ϕ-type over A. For any $\psi \in p$, since $\psi \to x = x$, we have $R^\phi(\psi) < R^\phi(x = x) = \omega$. Let $\psi(x)$ be a formula of minimal R^ϕ-rank in p, say $R^\phi(\psi(x)) = k < \omega$.

Now for any tuple b from A, if $\phi(x,b) \in p$, then $\psi(x) \wedge \phi(x,b) \in p$, so $R^\phi(\psi(x) \wedge \phi(x,b)) \geq k$ by minimality.

On the other hand, if $\phi(x,b) \notin p$, then $\neg\phi(x,b) \in p$, so $R^\phi(\psi(x) \wedge \neg\phi(x,b)) \geq k$ by the argument above. Suppose also that $R^\phi(\psi(x) \wedge \phi(x,b)) \geq k$. Then $\phi(x,b)$ and $\neg\phi(x,b)$ witness that $R^\phi(\psi(x)) \geq k+1$, which is a contradiction, since we already know that its rank is k. We conclude that if $\phi(x,b) \notin p$, then $R^\phi(\psi(x) \wedge \phi(x,b)) < k$.

Now the property $R^\phi(\psi(x) \wedge \phi(x,b)) \geq k$ is equivalent to the ability to build a ϕ-tree of heigh k whose leaves satisfy $\psi(x) \wedge \phi(x,b)$. This is a definable property of b with parameters those elements of A appearing in ψ. So p is definable over A.

(5) $\Rightarrow$ (3): Let $\kappa = |T|$, and let A be a set with $|A| \leq \kappa$. Each type in $S^\phi_x(A)$ is uniquely determined by its definition over A. But there are at most $|A| \cdot |T| = \kappa$ inequivalent formulas over A, so $|S^\phi_x(A)| \leq \kappa$, and ϕ is stable in κ. □

Theorem 2 (T is stable)**.** The following are equivalent:

(1) Every formula $\phi(x,y)$ is stable.

(2) T is stable in all κ such that $\kappa^{|T|} = \kappa$.

(3) T is stable in some κ.

Proof.

$$\begin{tikzcd} (1) \arrow[r, Rightarrow] & (2) \arrow[d, Rightarrow] \\ & (3) \arrow[ul, Rightarrow] \end{tikzcd}$$

(1) $\Rightarrow$ (2): Let κ be such that $\kappa^{|T|} = \kappa$, and let A be a set of cardinality at most κ. There is an injective map $S_x(A) \hookrightarrow \prod_{\phi(x,y)} S^\phi_x(A)$, associating to each map its collection of restrictions to ϕ-types for each formula $\phi(x,y)$. Now $|S_x(A)| \leq |\prod_{\phi(x,y)} S^\phi_x(A)| \leq \kappa^{|T|} = \kappa$, since $|S^\phi_x(A)| \leq \kappa$ for each ϕ (ϕ is stable, hence stable in κ). So T is stable in κ.

(2) $\Rightarrow$ (3): The only thing to note here is that *some* cardinal κ has $\kappa^{|T|} = \kappa$. But if λ is any cardinal, then taking $\kappa = \lambda^{|T|}$, we have $\kappa^{|T|} = (\lambda^{|T|})^{|T|} = \lambda^{|T| \cdot |T|} = \lambda^{|T|} = \kappa$.

(3) $\Rightarrow$ (1): Suppose T is stable in κ. Let $\phi(x,y)$ be a formula, and let A be a set of cardinality at most κ. There is a surjective map $S_x(A) \to S^\phi_x(A)$ given by restricting a type to its ϕ-formulas (surjective because any ϕ-type may be extended to a complete type by taking the complete type of a realization). So $|S^\phi_x(A)| \leq |S_x(A)| \leq \kappa$, and ϕ is stable in κ, hence stable. □

Theorem 3 (T is superstable)**.** The following are equivalent (for stable theories):

(1) For any tuple c and set A, there exists $B \subseteq_{\text{fin}} A$ with $c \mathop{\smile\hskip -0.9em ^| \ }_B A$.

(2) There is no infinite forking chain of finite subsets. That is, there is no tuple c and chain $A_0 \subset A_1 \subset \ldots$ of finite subsets such that $c \not\mathop{\smile\hskip -0.9em ^| \ }_{A_i} A_{i+1}$ for all i.

(3) For all complete types p, $U(p) < \infty$.

(4) T is stable in all $\kappa \geq 2^{|T|}$.

(5) T is stable in some κ such that $\kappa^{|T|} > \kappa$.

Note: (5) $\Rightarrow$ (2) requires countable language! I would like to know if (4) $\Rightarrow$ (2).

Proof.

$$\begin{tikzcd}
(3) \arrow[r, Rightarrow] & (1) \arrow[r, Rightarrow] \arrow[d, Rightarrow] & (4) \arrow[d, Rightarrow] \arrow[dl, dotted, "?"'] \\
& (2) \arrow[ul, Rightarrow] & (5) \arrow[l, dashed, Rightarrow]
\end{tikzcd}$$

(1) $\Rightarrow$ (2): Suppose for contradiction that we have a tuple c and chain $A_0 \subset A_1 \subset \dots$ as in the statement. Let $A = \bigcup_{i \in \omega} A_i$. By hypothesis, there is a finite subset $B \subset_{\text{fin}} A$ such that $c \mathop{\downarrow}_B A$. Now $B \subseteq A_n$ for some n. By monotonicity in the base, $c \mathop{\downarrow}_{A_n} A$, and by monotonicity on the right, $c \mathop{\downarrow}_{A_n} A_{n+1}$, contradiction.

(2) $\Rightarrow$ (3): I first claim (assuming only stability) is that there is an ordinal α such that for any finite set B and any type p over B, if $U(p) \geq \alpha$, then $U(p) = \infty$. Indeed, note that U-rank is preserved under automorphisms, so $U(\text{tp}(c/B))$ depends only on $\text{tp}(cB/\emptyset)$, the type of a finite tuple over the empty set. Since the collection of all types in tuples of finite length over the empty set forms a set, the collection of ordinals appearing as $U(\text{tp}(c/B))$ for B finite is a set. Take α to be their supremum.

Next, I claim (again assuming only stability) that if B is finite and $U(\text{tp}(c/B) = \infty$, then there is a finite $B' \supset B$ with $c \mathop{\not\downarrow}_B B'$ and $U(\text{tp}(c/B')) = \infty$. Indeed, taking α as in the first claim, $U(\text{tp}(c/B)) \geq \alpha + 1$. This is witnessed by a set $\widetilde{B} \supset B$ with $c \mathop{\not\downarrow}_B \widetilde{B}$ and $U(\text{tp}(c/\widetilde{B})) \geq \alpha$. Now by finite character, there is a finite tuple b' from $\widetilde{B}$ with $c \mathop{\not\downarrow}_B Bb'$. And restricting $\text{tp}(c/\widetilde{B})$ to $B' = Bb'$ can only make the U-rank go up, so $U(\text{tp}(c/B')) \geq \alpha$, and hence $U(\text{tp}(c/B')) = \infty$ since B' is finite.

Now we'll prove the contrapositive of (2) $\Rightarrow$ (3). Suppose that there is some type p over A with $U(p) = \infty$. Take c realizing p. Let $A_0 = \emptyset$. Restricting a type to a smaller set can only make the U-rank go up, so $U(\text{tp}(c/A_0)) \geq U(\text{tp}(c/A)) = \infty$. Applying the second claim repeatedly, we obtain a forking chain of finite sets $A_0 \subset A_1 \subset \dots$ with $U(\text{tp}(c/A_i)) = \infty$ and $c \mathop{\not\downarrow}_{A_i} A_{i+1}$ for all i.

(3) $\Rightarrow$ (1): Let B be the finite subset of A minimizing $U(\text{tp}(c/B)) < \infty$. Suppose $c \mathop{\not\downarrow}_B A$. Then $c \mathop{\not\downarrow}_B a$ for some finite tuple a from A. But then $U(\text{tp}(c/Ba)) < U(\text{tp}(c/B))$, contradicting minimality. So $c \mathop{\downarrow}_B A$.

(1) $\Rightarrow$ (4): Take $\kappa \geq 2^{|T|}$ and A a set of size at most κ. Given any type $p \in S_x(A)$ with a realization c, we have $B \subseteq_{\text{fin}} (A)$ with $c \mathop{\downarrow}_B A$. Then p is determined by its definition over $\text{acl}^{\text{eq}}(B)$, given by one defining formula over $\text{acl}^{\text{eq}}(B)$ for each formula in the language.

Now we count. There are at most κ many finite subsets of A. For each such subset B, $\text{acl}^{\text{eq}}(B)$ has size at most $|T|$, since each formula with parameters from B may add at most finitely many members to $\mathrm{acl}^{\mathrm{eq}}(B)$, so there are at most $|T|$ formulas with parameters from $\mathrm{acl}^{\mathrm{eq}}(B)$. A definition is a choice of one defining formula for each formula, so there are at most $|T|^{|T|} = 2^{|T|}$ definitions over $\mathrm{acl}^{\mathrm{eq}}(B)$. Then there are at most $\kappa \cdot 2^{|T|} = \kappa$ many types, and T is stable in κ.

(4) $\Rightarrow$ (5): The only thing to note here is that there is a cardinal $\kappa \geq 2^{|T|}$ such that $\kappa^{|T|} > \kappa$. Take any cardinal κ of cofinality ω (there are arbitrarily large such cardinals). Then $\kappa^{|T|} \geq \kappa^{\mathrm{cf}(\kappa)} > \kappa$.

(5) $\Rightarrow$ (2): (Countable language assumed!) We will prove the contrapositive. Suppose there is a tuple c and chain of finite sets $A_0 \subset A_1 \subset \dots$ with $c \not\mathrel{\raise0.2ex\hbox{$\smile$}\mkern-14mu\downarrow}_{A_i} A_{i+1}$ for all i. We will show that for all κ with $\kappa^\omega > \kappa$, T is unstable in κ.

We will produce a κ-branching ω-height tree of finite sets $\langle A_\eta \rangle_{\eta \in \kappa^{<\omega}}$ (view elements of $\kappa^{<\omega}$ as finite sequences from κ) by induction. One of the things we will ensure is that if η has length n ($l(\eta) = n$), $A_\eta \equiv A_n$. Take $A_{\langle\rangle} = A_0$.

Given $A_\eta \equiv A_{l(\eta)}$ for all η with η of length $\leq n$, and taking a particular η of length n, we will define $A_{\eta\alpha}$ for $\alpha \in \kappa$ by induction. The idea is to pick $A_{\eta\alpha}$ so that $\mathrm{tp}(A_{\eta\alpha}/A_\eta)$ matches $\mathrm{tp}(A_{n+1}/A_n)$ and so that $A_{\eta\alpha}$ is free over A_η from everything picked so far.

Formally, since $A_\eta \equiv A_n$, we can push forward $\mathrm{tp}(A_{n+1}/A_n)$ under an automorphism taking A_n to A_η, obtaining a type over A_η, take a nonforking extension of this type to $\widetilde{A} = (\bigcup_{\eta' : l(\eta') \leq n} A_\eta) \cup (\bigcup_{\eta\beta : \beta < \alpha} A_\beta)$, and pick $A_{\eta\alpha}$ realizing this type, so $A_{\eta\alpha} \mathrel{\raise0.2ex\hbox{$\smile$}\mkern-14mu\downarrow}_{A_\eta} \widetilde{A}$.

Given this construction, let $A^* = \bigcup_{\eta \in \kappa^{<\omega}} A_\eta$. Since all η are finite, $|A^*| \leq |\kappa^{<\omega}| = \kappa$. But we will associate distinct type p_σ over A^* to each path through the tree $\sigma \in \kappa^\omega$. Since $\kappa^\omega > \kappa$, this will show that T is unstable in κ.

Let $A = \bigcup_n A_n$ (the union of the original forking chain), and let $p = \mathrm{tp}(c/A)$. Note that $p \upharpoonright A_{n+1}$ forks over A_n for all n.

Let $\sigma \in \kappa^\omega$. For all $n \in \omega$, let $A_n^\sigma = A_{\sigma\upharpoonright n}$. Then we have a chain $A_0^\sigma \subset A_1^\sigma \subset \dots$. Let $A^\sigma = \bigcup_n A_n^\sigma$. Since $A_n^\sigma \equiv A_n$ for all n, $A^\sigma \equiv A$. Let $\widehat{p_\sigma}$ be the pushforward of p to a type over A^σ by an automorphism moving A to A^σ, and let p_σ be a nonforking extension of $\widehat{p_\sigma}$ to a type over A^*. Note that $p_\sigma \upharpoonright A_{n+1}^\sigma = \widehat{p_\sigma} \upharpoonright A_{n+1}^\sigma$ forks over A_n^σ for all n.

It remains to show that for $\sigma \neq \tau$, $p_\sigma \neq p_\tau$. Let $n+1$ be the first place where the sequences σ and τ differ. So $\sigma \upharpoonright n = \tau \upharpoonright n$, call it η. Then $A_n^\sigma = A_n^\tau = A_\eta$. Since $p_\tau \upharpoonright A_{n+1}^\tau$ forks over A_η, it suffices to show that $p_\sigma \upharpoonright A_{n+1}^\tau$ does not fork over A_η.

Claim: For all $N > n$, $A_{n+1}^\tau \mathrel{\raise0.2ex\hbox{$\smile$}\mkern-14mu\downarrow}_{A_\eta} A_N^\sigma$.

The proof is by induction on N. In the base case, we need that $A_{n+1}^\tau \mathrel{\raise0.2ex\hbox{$\smile$}\mkern-14mu\downarrow}_{A_\eta} A_{n+1}^\sigma$. Let $\sigma \upharpoonright n+1 = \eta\alpha$ and let $\tau \upharpoonright n+1 = \eta\beta$. Without loss of generality (by symmetry of forking), $\alpha < \beta$, and we chose $A_{\eta\beta}$ free from $A_{\eta\alpha}$ over A_η for all $\alpha < \beta$.

Now assume we have $A_{n+1}^\tau \mathrel{\raise0.2ex\hbox{$\smile$}\mkern-14mu\downarrow}_{A_\eta} A_N^\sigma$ with $N \geq n+1$. We chose A_{N+1}^σ free from all $A_{\eta'}$ with $l(\eta') \leq N$ over A_N^σ, and in particular, $A_{N+1}^\sigma \mathrel{\raise0.2ex\hbox{$\smile$}\mkern-14mu\downarrow}_{A_N^\sigma} A_{n+1}^\tau$. By symmetry, $A_{n+1}^\tau \mathrel{\raise0.2ex\hbox{$\smile$}\mkern-14mu\downarrow}_{A_N^\sigma} A_{N+1}^\sigma$, and by transitivity, applying the induction hypothesis, $A_{n+1}^\tau \mathrel{\raise0.2ex\hbox{$\smile$}\mkern-14mu\downarrow}_{A_\eta} A_{N+1}^\sigma$, as desired.

As a consequence of the claim, $A_{n+1}^\tau \mathrel{\raise0.2ex\hbox{$\smile$}\mkern-14mu\downarrow}_{A_\eta} A^\sigma$. Indeed, if $A_{n+1}^\tau \not\mathrel{\raise0.2ex\hbox{$\smile$}\mkern-14mu\downarrow}_{A_\eta} A^\sigma$, then there is a finite tuple b from A^σ with $A_{n+1}^\tau \not\mathrel{\raise0.2ex\hbox{$\smile$}\mkern-14mu\downarrow}_{A_\eta} b$, and b is contained in A_N^σ for some N, so $A_{n+1}^\tau \not\mathrel{\raise0.2ex\hbox{$\smile$}\mkern-14mu\downarrow}_{A_\eta} A_N^\sigma$, contradicting the claim.

Now let c_σ realize p_σ. Since p_σ is a nonforking extension of $\widehat{p_\sigma} = p_\sigma \restriction A^\sigma$, $c_\sigma \downarrow_{A^\sigma} A^*$. By monotonicity, $c_\sigma \downarrow_{A^\sigma} A^\tau_{n+1}$, so $A^\tau_{n+1} \downarrow_{A^\sigma} c_\sigma$ by symmetry. But also $A^\tau_{n+1} \downarrow_{A_\eta} A^\sigma$. By transitivity, $A^\tau_{n+1} \downarrow_{A_\eta} c_\sigma$, and by symmetry again, $c_\sigma \downarrow_{A_\eta} A^\tau_{n+1}$, that is to say, $p_\sigma \restriction A^\tau_{n+1}$ does not fork over A_η, as desired. □

Theorem 4 (T is totally transcendental)**.** The following are equivalent:

(1) $\mathrm{RM}(x = x) < \infty$

(2) T is stable in all $\kappa \geq |T|$

(3) T is stable in some $\kappa < 2^{|T|}$

Note: (3) $\Rightarrow$ (1) requires countable language! I would like to know if (2) $\Rightarrow$ (1).

Proof.

$$\begin{tikzcd} (1) \arrow[r, Rightarrow] & (2) \arrow[l, dotted, bend right, "?"'] \arrow[d, Rightarrow] \\ & (3) \arrow[ul, Rightarrow, dashed] \end{tikzcd}$$

(1) $\Rightarrow$ (2): We'll prove the contrapositive. Suppose T is unstable in some $\kappa \geq |T|$. Let A be a set of size at most κ with $|S_x(A)| > \kappa$. Given a formula $\phi(x)$ with parameters from A, write U_ϕ for the set of all types is $S_x(A)$ containing ϕ.

Claim: Given any formula $\phi(x)$ with $|U_\phi| > \kappa$, there is a formula $\psi(x)$ such that $|U_{\phi\wedge\psi}| > \kappa$ and $|U_{\phi\wedge\neg\psi}| > \kappa$.

Suppose not. Then there is a formula ϕ with $|U_\phi| > \kappa$ such that for all $\psi(x)$, $|U_{\phi\wedge\psi}| \leq \kappa$ or $|U_{\phi\wedge\neg\psi}| \leq \kappa$. Note that it cannot be both, since $U_\phi = U_{\phi\wedge\psi} \cup U_{\phi\wedge\neg\psi}$. Let Ψ be the set of all formulas $\psi(x)$ such that $|U_\psi| \leq \kappa$ (so Ψ contains exactly one of ψ and $\neg\psi$ for each ψ). Now we can write $U_\phi = (\bigcup_\Psi U_{\phi\wedge\psi}) \cup (\bigcap_\Psi U_{\phi\wedge\neg\psi})$, since a type contains some formula in Ψ or the negations of all formulas in Ψ. But $\bigcap_\Psi U_{\phi\wedge\neg\psi}$ contains at most one type, since a type in the intersection is determined entirely as $\{\neg\psi \mid \psi \in \Psi\}$.

So $|U_\phi| = |\,(\bigcup_\Psi U_{\phi\wedge\psi}) \cup (\bigcap_\Psi U_{\phi\wedge\neg\psi})\,| \leq |T| \cdot \kappa + 1 = \kappa$, contradiction.

Now we will use the claim to build a tree which will contradict $\mathrm{RM}(x = x) < \infty$. For all $\eta \in 2^{<\omega}$ (view elements of $2^{<\omega}$ as finite binary sequences), define a formula ϕ_η by induction such that $|U_{\phi_\eta}| > \kappa$ for all η. Let $\phi_{\langle\rangle}$ be $x = x$. By hypothesis, $|U_{x=x}| = |S_x(A)| > \kappa$.

Given ϕ_η with $|U_{\phi_\eta}| > \kappa$, apply the claim to obtain ψ with $|U_{\phi_\eta\wedge\psi}| > \kappa$ and $|U_{\phi_\eta\wedge\neg\psi}| > \kappa$. Let $\phi_{\eta 0} = \phi_\eta \wedge \psi$ and $\phi_{\eta 1} = \phi_\eta \wedge \neg\psi$.

I will argue by induction that every formula in the tree has Morley rank at least α for all ordinals α.

Base case: Each formula ϕ_η has $|U_{\phi_\eta}| > \kappa$, so ϕ_η is consistent, and $\mathrm{RM}(\phi_\eta) \geq 0$.

Limit case: By induction, $\mathrm{RM}(\phi_\eta) \geq \alpha$ for all $\alpha < \lambda$ a limit, so $\mathrm{RM}(\phi_\eta) \geq \lambda$.

Successor case: We have $\mathrm{RM}(\phi_\eta) \geq \alpha$ for all η. Now the formulas $\phi_{\eta 1}$, $\phi_{\eta 01}$, $\phi_{\eta 001}$, … all imply ϕ_η, are pairwise contradictory, and have $\mathrm{RM}(\phi_\eta \wedge \phi_{\eta 0\ldots 01}) = \mathrm{RM}(\phi_{\eta 0\ldots 01}) \geq \alpha$ by induction. So they witness $\mathrm{RM}(\phi_\eta) \geq \alpha + 1$.

Now since $x = x$ is the root of the tree, $\mathrm{RM}(x = x) = \infty$, as desired.

(2) $\Rightarrow$ (3): The only thing to note here is that T is stable in $|T| < 2^{|T|}$.

(3) $\Rightarrow$ (1): (Countable language assumed!) Again, we'll prove the contrapositive. Suppose $\mathrm{RM}(x = x) = \infty$. We'll show that T is unstable in every $\kappa < 2^\omega$. To do this, it suffices to find a countable set of parameters A such that $S_x(A) = 2^\omega$.

Since Morley rank corresponds to Cantor-Bendixson rank on the type space $S = S_x(M)$ for a sufficiently saturated model M, $\mathrm{RM}(x = x) = \infty$ means that S^∞, the final derived set of S (which has no isolated points), is not empty. For all $\eta \in 2^{<\omega}$, define a formula ϕ_η by induction such that $U_{\phi_\eta} \cap S^\infty \neq \emptyset$.

Let $\phi_{\langle\rangle}$ be $x = x$. Given ϕ_η with $U_{\phi_\eta} \cap S^\infty \neq \emptyset$, pick two types in this intersection, p and q. There cannot be just one, or it would be isolated in S^∞ by ϕ_η. Let $\phi_{\eta 0}$ be a formula separating them and let $\phi_{\eta 1}$ be its negation, so $\phi_{\eta 0} \in p$ and $\phi_{\eta 1} \in q$. Then since $p \in U_{\phi_{\eta 0}} \cap S^\infty$ and $q \in U_{\phi_{\eta 1}} \cap S^\infty$, these intersections are not empty.

Let A be the set of all parameters appearing in the formulas ϕ_η. A is countable, since there are countably many such formulas. But $|S_x(A)| \geq 2^\omega$, since each path through the tree $\eta \in 2^\omega$ gives rise to a distinct consistent type by compactness. $\square$

Corollary (Stability spectrum for countable theories)**.** If T is countable, there are just four possibilities for the stability spectrum of T:

(1) St-Spec$(T) = \emptyset$ (T is unstable).

(2) St-Spec$(T) = \{\kappa \mid \kappa^\omega = \kappa\}$ (T is stable but not superstable).

(3) St-Spec$(T) = \{\kappa \mid \kappa \geq 2^\omega\}$ (T is superstable but not totally transcendental).

(4) St-Spec$(T) = \{\kappa \mid \kappa \geq \omega\}$ (T is totally transcendental, also called ω-stable).