RESEARCH STATEMENT

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What makes some problems easier to solve than others? What makes some areas of mathematics easier to understand than others? Much of mathematical logic is concerned with variants of these broad questions. Computer scientists classify decidable problems by their computational complexity, reverse mathematicians measure non-constructive theorems by the strength of the comprehension and choice principles needed to prove them, and set theorists stratify axioms which go beyond ZFC set theory by their consistency strength.

Model theorists, on the other hand, look for dividing lines in the complexity of theories of mathematical structures in first-order logic, in an attempt to separate "tame" theories (e.g. the theory of vector spaces or algebraically closed fields) from "complex" ones (e.g. Peano arithmetic or ZFC set theory), and to develop general techniques for studying the tame theories and their models (a *model* for a theory T is just a mathematical structure satisfying the axioms of T). These tools allow us to understand definability (sets, relations, and functions definable by first-order formulas) in tame structures (e.g. the ordered group of integers, the real field, or the complex field), which can be applied to purely mathematical questions about these structures.

Modern model theory is heavily influenced by Saharon Shelah's landmark work on classification theory [23]. Shelah developed a deep structure theory for the class of *stable* first-order theories and the models of these theories. Though very abstract, Shelah's work (and later developments in stability theory, by Ehud Hrushovski, Anand Pillay, and many others) is the key component in many of the most striking applications of model theory, such as Hrushovki's proof of the Mordell–Lang conjecture for function fields [9].

The *neostability* program seeks to generalize Shelah's work to other dividing lines beyond stability. There are a host of these properties (see [20] for a "map"), and unfortunatley they are often named by arcane acronyms like $NSOP_1$ or NTP_2. But as the theory develops, evidence accrues that some dividing lines are especially important and robust, demonstrated by structure theorems for theories on the "tame" side of the line and non-structure theorems for theories on the "complex" side. This program has been very successful for several classes of theories, most notably the simple theories (see [27]), the NIP theories (see [24]), and the o-minimal theories (see [26]).

In my work, I am particularly interested in *generic* and *random* structures, especially in relation to the neostability program described above. By generic, I usually mean *existentially closed*; this is a richness condition on a model M which says that any first-order formula without quantifiers that has a solution in a larger model already has a solution in M. For example, a generic field is algebraically closed, and a generic linear order is dense without endpoints. When the generic models of a theory T can themselves be axiomatized by a first-order theory T^*, we call T^* the *model companion* of T.

The generic models of T are often more amenable to model-theoretic analysis than arbitrary models of T, and studying them can shed light on the entire class of models of T, especially when the model companion T^* exists (this is a generalization of the idea that

it is useful in field theory to embed a field in its algebraic closure). Abraham Robinson already pioneered this approach to applied model theory in the 1950s, and it continues to bear fruit. For example, technical results about the stable theory DCF (the generic theory of fields equipped with a derivation) and the simple theory ACFA (the generic theory of fields equipped with a distinguished automorphism) have been central to the applications of model theory to arithmetic geometry alluded to above (e.g. [9] and [10]).

Another sense of the word generic refers to Fraïssé limits. If $\mathcal{K}$ is a class of finite structures (think of the class of finite graphs) satisfying certain hypotheses, then there is a countably infinite structure $M_{\mathcal{K}}$, the Fraïssé limit of $\mathcal{K}$, which is universal (every structure in $\mathcal{K}$ embeds into it) and homogeneous (any two such embeddings are conjugate by an automorphism of $M_{\mathcal{K}}$). If we look at the space $X_{\mathcal{K}}$ of all structures M with domain $\mathbb{N}$, such that all finite substructures of M are in $\mathcal{K}$, then the isomorphism class of the Fraïssé limit $M_{\mathcal{K}}$ is *comeager* in $X_{\mathcal{K}}$: generic from the point of view of view of Baire category. Fraïssé limits are at the heart of many connections between model theory and combinatorics, descriptive set theory, and permutation group theory (see [4] and [13]).

If $\mathcal{G}$ is the class of the class of finite graphs, the Fraïssé limit $M_{\mathcal{G}}$ is known as the *Rado graph*, or the *random graph*. The Rado graph also arises from the Erdős–Rényi random graph construction: fix a countably infinite vertex set and a probability $0 < p < 1$, and put an edge between each pair of distinct vertices independently with probability p. The resulting graph is isomorphic to the Rado graph with probability 1.

This random construction can be formalized as a probability measure μ on the space $X_{\mathcal{G}}$, which is moreover invariant and ergodic for the natural group action of S_{∞} on this space (this group action is sometimes called the "logic action", and its orbits are exactly the isomorphism classes). It turns out that this kind of measure on $X_{\mathcal{G}}$, which we call an *ergodic structure*, encodes exactly the same information as a *graphon*, a limit structure for a sequence of finite graphs which converges in the appropriate sense (see [19]), and its natural generalization to other spaces $X_{\mathcal{K}}$ can be viewed as "generalized graphons" for other classes of structures $\mathcal{K}$.

The Baire category / measure analogy between Fraïssé limits and ergodic structures was a major theme of my PhD thesis [15]. In a joint paper [1] with Nate Ackerman, Cameron Freer, and Rehana Patel that came out of that thesis, we characterized those ergodic structures which (unlike the Erdős–Rényi measure) do not give measure 1 to any single isomorphism class. The proof used a detailed model-theoretic "Morley–Scott analysis" of ergodic structures, providing evidence that these measures can be profitably viewed as *random* analogues of ordinary structures.

In the rest of this statement, I will describe four major themes in my current research:

(1) *Generic structures and NSOP$_1$.*
(2) *Random structures, Fraïssé limits, and zero-one laws.*
(3) *Reductions that preserve genericity.*
(4) *Other logics (and connections with computer science).*

The first three sections share the common thread of generic and random structures in model theory. The last section can be read independently from the rest of the statement.

Generic structures and NSOP$_1$: One dividing line in the neostability hierarchy, called NSOP$_1$, has been the subject of increased attention recently. This is largely due to work of Artem Chernikov, Itay Kaplan, and Nick Ramsey [7, 12], who showed that NSOP$_1$ theories can be characterized by the existence of an abstract notion of independence called

Kim independence, which is a generalization of forking independence in simple and stable theories.

One of the most interesting features of NSOP_1 is its apparent robustness under "generic constructions". For example, in [18], Ramsey and I showed that the generic expansion of an NSOP_1 theory by Skolem functions, or by new constant, function, or relation symbols interpreted arbitrarily, is NSOP_1. A specific example is the generic theory of all $\mathcal{L}$-structures in an arbitrary language, which was previously known to be simple when $\mathcal{L}$ is relational. Using a result from Peter Winkler's thesis [28] on the existence of the generic Skolemization, it follows that any NSOP_1 theory which eliminates the quantifier $\exists^\infty$ has an expansion to an NSOP_1 theory with built-in Skolem functions.

In joint work with Gabe Conant [6], we showed that the theory of generic projective planes is NSOP_1 (a projective plane is an incidence structures in which any two points are incident with a unique line and any two lines are incident with a unique point), and we generalized this to structures we call (m, n)-pseudoplanes (incidence structures in which any m points are incident with $(n-1)$ lines and any n lines are incident with $(m-1)$ points). These examples are particularly interesting, because viewed as generic bipartite graphs omitting $K_{m,n}$, they are bipartite analogues of the generic K_n-free graphs, which are the canonical examples of SOP_3 but NSOP_4 theories.

I continued this work with an undergraduate student, Matisse Peppet, in a successful summer REU project. She investigated the combinatorics of (m, n)-pseudoplanes, identified a robust notion of non-degeneracy for these structures, and showed that if $n \geq 2$ and $m \geq 3$, then every non-degenerate (m, n)-psuedoplane is infinite. In particular, her work implies that in these cases, the theory of generic (m, n)-pseudoplanes has no prime model. The world of generic combinatorial structures is full of interesting but very concrete problems, and look forward to working with undergraduates on similar projects in the future.

In forthcoming joint work with Minh Tran and Erik Walsberg, we have investigated interpolative fusions: Given a family of languages $(\mathcal{L}_i)$ with common intersection $\mathcal{L}_\cap$ and a family $(T_i)_{i \in I}$ of model-complete $\mathcal{L}_i$-theories, with a common set $T_\cap$ of $\mathcal{L}_\cap$-consequences, the interpolative fusion is the model companion of the union $\bigcup_{i \in I} T_i$ (if it exists). Interpolative fusions provide a unified framework for studying a wide variety of examples of generic theories in model theory, some of which (e.g. algebraically closed fields with multiple independent valuations) are explicitly interpolative fusions, while others (e.g. DCF and ACFA) are bi-interpretable with interpolative fusions.

In this work, along with providing sufficient conditions for the existence of the interpolative fusion and proving quantifier-elimination results, we showed that under appropriate hypotheses on $T_\cap$ (including stability), if all T_i are NSOP_1, then the interpolative fusion is NSOP_1. This gives an extremely flexible recipe for producing new NSOP_1 theories, which can be used to motivate and test conjectures in this area.

All of this work suggests a very interesting problem in neostability: is there a class of theories which is similarly robust under generic constructions, but which also contains ordered structures? One motivation comes from NTP_2, a class of theories containing the simple theories, but also some theories with the strict order property, such as dense linear orders and p-adic fields.

Problem 1. Complete the analogy: simplicity is to NTP_2 as NSOP_1 is to X. Develop a theory of Kim independence in the context of theories with property X.

The class of theories with the conjectural property X should include the NTP_2 theories and the NSOP_1 theories, and thus a solution to this problem would provide the broadest

class of theories for which a satisfying theory of independence has been developed to date. To make the problem more concrete, we know of several natural examples of first-order theories which should satisfy property X, but which are not on the tame side of any other known model-theoretic dividing lines: e.g. the generic theory of parameterized linear orders, the generic Skolemization of $(\mathbb{Q}, \leq)$, and Tran's theory [25] of an algebraically closed field F of finite characteristic with cyclically ordered multiplicative group induced by a character $F^{\times} \to \mathbb{C}^{\times}$. These theories will serve as important test cases.

Random structures, Fraïssé limits, and zero-one laws: Above, I discussed how the Rado graph can be obtained as the Fraïssé limit of the class $\mathcal{G}$ of finite graphs and also (with probability 1) from the Erdős–Rényi random construction. In fact, it arises in a third way, from the zero-one law for finite graphs. For every first-order sentence φ in the language of graphs, the limit of the fraction of graphs of size n satisfying φ is 0 or 1 as $n \to \infty$. And the Rado graph is the unique countable model of the set of all sentences with limiting probability 1. In particular, it is pseudofinite: every sentence in its theory has a finite model.

A major open problem in combinatorial model theory is the question of whether the generic triangle-free graph (which has SOP_3, hence is not simple or NSOP_1) is pseudofinite. In [14], I was able to show pseudofiniteness for several Fraïssé limits with NSOP_1 theories. This led me to raise the following question, which seems very hard.

Question 2. For every pseudofinite Fraïssé limit $M_{\mathcal{K}}$, is $\mathrm{Th}(M_{\mathcal{K}})$ NSOP_1?

In joint work in progress with Cameron Hill, we are working on more tractable versions of the open problems above. In [14] (which is a chapter of my PhD thesis [15]), I showed that if a Fraïssé class $\mathcal{K}$ has disjoint n-amalgamation for all n (these are "higher dimensional" analogues of the amalgamation property), then there is a sequence of measures $(\mu_k)_{k \in \mathbb{N}}$ on the space of structures in $\mathcal{K}$ with domain $[k]$, which converge (in the appropriate sense) to an ergodic structure μ, such that μ gives measure 1 to the isomorphism class of the Fraïssé limit $M_{\mathcal{K}}$, and the μ_k have a zero-one law converging to the theory of $M_{\mathcal{K}}$. If $\mathcal{K}$ admits such a sequence of measures, I call the theory $\mathrm{Th}(M_{\mathcal{K}})$ *strongly pseudofinite*. In the case of the class $\mathcal{G}$ of finite graphs, the uniform measures on the spaces of graphs with domain $[k]$ witness that the theory of the Rado graph is strongly pseudofinite.

Conjecture 3. Let $\mathcal{K}$ be a Fraïssé class such that $\mathrm{Th}(M_{\mathcal{K}})$ is strongly pseudofinite. Then:
(A) $\mathrm{Th}(M_{\mathcal{K}})$ is simple (with SU-rank 1).
(B) $M_{\mathcal{K}}$ has a reduct of a Fraïssé limit $M_{\mathcal{K}'}$, such that the Fraïssé class $\mathcal{K}'$ has disjoint n-amalgamation for all n.

Either of these conjectures would establish that the generic triangle free graph is not strongly pseudofinite. At least (A) seems within reach.

Reductions that preserve genericity: As another instance of the problem of measuring complexity in mathematics, descriptive set theorists study certain kinds of reductions between analytic equivalence relations on Polish spaces. A canonical example of such a relation is the isomorphism relation $\cong_T$ on the space X_T of models of a theory T with domain $\mathbb{N}$. The relation $\cong_T$ is in fact an orbit equivalence relation: the isomorphism classes are the orbits of the logic action of S_{∞} on X_T. A reduction from $\cong_T$ to $\cong_{T'}$ is then a way of encoding models of T as models of T' in a way which preserves isomorphism. Usually some condition on the definability of the encoding is enforced; e.g. the reduction function is required to be Borel or at least Baire measurable.

In recent work with Aristotelis Panagiotopoulos [17], we study a kind of reduction we call ∗-reductions: these are Baire measurable reductions which preserve generic properties, in the sense that the preimage of a comeager set is comeager. We introduce a new kind of dynamical obstruction to reductions, and present an infinite family of orbit equivalence relations, none of which is ∗-reducible to any other.

The key observation is that the topology of an orbit equivalence relation encodes more than just the equivalence relation: it also allows us to define a directed graph, the Becker graph, whose vertices are the equivalence classes. In the case of a logic action, this graph is the embeddability relation between structures. Our main theorem is that if $f \colon (X, E) \to (Y, F)$ is a ∗-reduction, then there is an invariant comeager subset C of X and an invariant non-meager subset D of Y such that the Becker graphs on C and D are isomorphic.

Using this theorem, we are able to demonstrate non-reducibility for a family of classes of structures $(\mathcal{B}_n)_{n \in \omega}$, which were originally defined in [2] by John Baldwin, Martin Koerwien, and Chris Laskowski. These classes have the property that $\mathcal{B}_n$ has disjoint $(n-1)$-amalgamation, but no non-trivial instances of disjoint n-amalgamation. Roughly speaking, these differences in the "amalgamation dimension" are visible in the Becker graphs and provide obstructions to ∗-reducibility between the $\mathcal{B}_n$.

Currently, our techniques only apply to the embeddability relation between structures; that is, we look at Becker *graphs*, not Becker *categories*. In future work, we would like to develop functorial versions of these techniques, possibly in the setting of Polish groupoids.

Other logics (and connections with computer science): My other research has focused on non-first-order logics, largely motivated by connections with computer science.

First, my two papers with my postdoc mentor Larry Moss ([21] and [16]) are part of his long-term project on natural logics: logics with features which model natural language. These logics are typically much weaker than first-order logic; they are decidable, and sometimes even computationally tractable. To understand these logics, one needs to prove precise enough completeness theorems to determine the computational complexity of the consequence relation $\models$. We analyzed a particular family of natural logics and identified features which distinguish tractable logics ($\models$ is in P) from intractable logics ($\models$ is co-NP hard). These dichotomies have implications for automated reasoning from natural language.

In joint work with Siddharth Bhaskar, another postdoc at IU, we studied the neostability dividing lines from model theory in the context of least fixed point (inductive) logic over classes of finite structures [3]. In particular, we showed that in this setting, stability and NIP coincide. The study of inductive logic is motivated by notions of computation over finite data structures and is related to issues in finite model theory (specifically McColm's conjecture, which was settled by Yuri Gurevich, Neil Immerman, and Shelah [8]).

Finally, in current work, I am developing a categorical extension of first-order logic that replaces the variable contexts of formulas and the underlying sets of structures with objects in categories with sufficient structure, e.g. locally finitely presentable categories. By dualizing, I obtain a "cologic", which naturally extends semantics, notions, and methods of model theory to profinite structures and coalgebras. This unifies some earlier examples of model-theoretic methods for profinite structures in the literature (see [5] and [11]). And it promises to provide logical tools for working coalgebras, which are frequently used in computer science to model infinite data types (see [22]). I have a paper in preparation laying out the foundations, but there is much work to be done here developing the model theory.

In the future, I am very interested in collaborating with computer scientists on further developments in these projects, and on applications of logic to computer science.

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