# Independence in generic structures

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# What is this talk about?

Background:

- Model theory, classification theory, notions of independence.
- Generic structures / generic theories.
- The specific setting: $\text{NSOP}_1$.

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This is all joint work:

- Alex Kruckman and Nicholas Ramsey, *Generic expansion and Skolemization in $NSOP_1$ theories*, arXiv:1706.06616, June 2017.
- Gabriel Conant and Alex Kruckman, *Independence in generic incidence structures*, arXiv:1709.09626, September 2017.

# What is model theory about?

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Emphasis on **semantics**, rather than *syntax*:
**Models**, rather than *proofs*.
**Definable sets**, rather than *formulas*.
**Applications**, rather than *foundations*.

# Complicated theories

The first-order theories ZFC (set theory) and PA (Peano arithmetic) are very complicated:

- Interpret large chunks of mathematics.
- Undecidable, and no computably enumerable completions (Gödel's Theorems apply).
- Quantifier hierarchies do not collapse.
- Lots of complicated definable sets.
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- Lots of complicated models.

These theories make models theorists sad.
To study their model theory, very specialized tools are needed.

# Tame theories

The theories $\mathrm{ACF}_0$ (algebraically closed fields of characteristic $0$) and $\mathbb{Q}$-Mod (rational vector spaces) are very tame:

- Decidable and complete.
- Quantifier elimination.
- Definable sets are easy to understand and mathematically meaningful.
- Few models, classified up to isomorphism by transcendence degree over $\mathbb{Q}$ (in $\mathrm{ACF}_0$) and dimension (in $\mathbb{Q}$-Mod).

It follows that these theories are *uncountably categorical*: Up to isomorphism, there is only one model of size $\kappa$, for all $\kappa > \aleph_0$.

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These theories make model theorists happy.
How can we find theories which will make us happy?
What distinguishes tame theories from complicated ones?

# Generic theories: Robinson ('50s and '60s)

Question 1: How can we find theories which will make us happy?

**Fact**

*For a theory $T$, the following are equivalent:*

1. *$T$ can be axiomatized by sentences of the form $\forall\overline{x}\exists\overline{y}\,\varphi(\overline{x},\overline{y})$, where $\varphi$ is quantifier-free and $\overline{x}$ and $\overline{y}$ are possibly empty tuples.*
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## Definition

A model $N \models T$ is *existentially closed* if whenever $N \subseteq N' \models T$, if $N' \models \exists\overline{y}\,\varphi(\overline{a},\overline{y})$ with $\overline{a} \in N$ and $\varphi$ quantifier-free, then $N \models \exists\overline{y}\,\varphi(\overline{a},\overline{y})$.

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For example, a field $K$ is existentially closed if and only if it is algebraically closed: if a system of polynomial equations has a solution in a field extension $K \subseteq K'$, then it already has a solution in $K$ (Nullstellensatz).

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- Let $\mathcal{C} = \mathrm{Mod}(T)$ be the class of all models of $T$.
- Let $\mathcal{C}^* \subseteq \mathcal{C}$ be the subclass of existentially closed models.
- If there is a theory $T^*$ such that $\mathcal{C}^* = \mathrm{Mod}(T^*)$, then we call $T^*$ the *model companion* of $T$, or the *generic theory* of $\mathcal{C}$.

# Examples of generic theories

| Theory | Generic theory |
|---|---|
| Fields | Algebraically closed fields |
| Integral domains | Algebraically closed fields |
| Ordered fields | Real closed fields |
| Linear orders | Dense linear orders without endpoints |
| Boolean algebras | Atomless Boolean algebras |
| Graphs | The theory of the random (Rado) graph |
| Groups | None |
| Division rings | None |

# We like generic theories

Suppose $T^*$ is the model companion of $T$. Then:

- $T^*$ *eliminates quantifiers* (at least) down to existential formulas: Every formula $\varphi(\overline{x})$ is equivalent to $\exists \overline{y}\, \psi(\overline{x}, \overline{y})$ with $\psi$ quantifier-free.

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- Models of $T^*$ serve as *universal domains* for $\mathcal{C} = \text{Mod}(T)$: If $\mathbb{M}$ is a large saturated model ("monster model") of $T^*$, then it (elementarily) embeds all small models of $T^*$, and hence embeds all small models of $T$. We can make the simplifying assumption that all objects of interest sit inside the monster model.

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For example, the theory of fields can be completed to $\text{Th}(\mathbb{Q})$ which is *complicated*. But a completion of the theory of algebraically closed fields is determined by fixing the characteristic. These theories $\text{ACF}_p$ are *tame*.

# New ideas: Morley and Baldwin–Lachlan ('60s and '70s)

Question 2: What distinguishes tame theories from complicated ones?

**Theorem (Morley, '65)**

*If a countable theory $T$ is categorical in some uncountable cardinal $\kappa$ (there is only one model of $T$ of cardinality $\kappa$ up to isomorphism), then $T$ is categorical in every uncountable cardinal.*

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*Idea of proof (Baldwin–Lachlan, '71):*
Every such theory $T$ has a formula $\varphi(x)$ defining a *strongly minimal set*, which supports a "geometry" (a matroid), which has a *dimension*. Just as in the case of $\mathsf{ACF}_0$ and $\mathbb{Q}$-Mod, models of $T$ are classified up to isomorphism by their dimension.

Uncountably categorical theories are the tamest of the tame.

# Stable theories: Shelah ('70s and '80s)

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(1) Focus on *dividing lines*, often defined by combinatorial patterns in definable sets. Both sides of the line should have powerful implications for structure / nonstructure. The most important is stability.

**Definition**

$T$ is *stable* if no formula $\varphi(x; y)$ has the *order property*: there exist tuples $(a_i)_{i\in\omega}$ and $(b_j)_{j\in\omega}$ such that $\mathbb{M} \models \varphi(a_i; b_j) \iff i \leq j$.

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(2) Focus on *forking independence* defined globally on subsets of $\mathbb{M}$:

$$A \mathop{\smile\!\!\!\!\!\mid}^{f}_{C} B \text{ is read "} A \text{ is independent from } B \text{ over } C\text{"}.$$

Generalizes linear and algebraic independence in $\mathbb{Q}$-Mod and $\mathrm{ACF}_0$.
In some cases ($T$ *superstable*), $\mathop{\smile\!\!\!\!\!\mid}^{f}$ gives rise to geometries and dimensions.

# Example: algebraic independence

I'm not going to define forking independence ($\mathop{\smile\!\!\!\!\!\!|}^f$), but I'll define a simpler independence relation: algebraic independence ($\mathop{\smile\!\!\!\!\!\!|}^a$).

## Definition

A formula $\varphi(x;a)$ is *algebraic* if it has only finitely many solutions (in $\mathbb{M}$). The (model-theoretic) *algebraic closure* of $A$, $\mathrm{acl}(A)$, is the set of all elements $b \in \mathbb{M}$ which satisfy some algebraic formula with parameters from $A$.

We define $A \mathop{\smile\!\!\!\!\!\!|}^a_C B \iff \mathrm{acl}(AC) \cap \mathrm{acl}(BC) = \mathrm{acl}(C)$.

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- In $\mathbb{Q}$-Mod, $\mathrm{acl}(A)$ is the subspace spanned by $A$, and $\mathop{\smile\!\!\!\!\!\!\,|}^a$ is the usual linear independence.
- In ACF$_0$, $\mathrm{acl}(A)$ is the algebraic closure of the field generated by $A$, and $\mathop{\smile\!\!\!\!\!\!\,|}^a$ is the usual algebraic independence.
- In a general stable theory, we may have $\mathop{\smile\!\!\!\!\!\!\,|}^f \neq \mathop{\smile\!\!\!\!\!\!\,|}^a$, but in any theory, $A \mathop{\smile\!\!\!\!\!\!\,|}^f_C B \implies A \mathop{\smile\!\!\!\!\!\!\,|}^a_C B$.

# The main gap

The main result of *Classification Theory* is the following theorem.

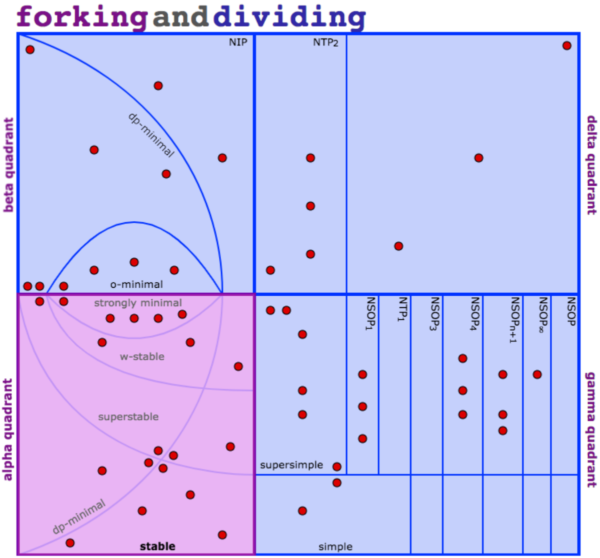
**Theorem (Shelah)**

*For every countable complete theory $T$, either:*

- *$T$ has the maximum number of models, $2^\kappa$, for all $\kappa > \aleph_0$, or*
- *($T$ is superstable, NDOP, NOTOP, and shallow.) The models of $T$ can be classified by certain "forking independent trees" of countable models. Various shapes of classification occur, and in each case, counting the possible trees yields precise descriptions of the number of models in all uncountable cardinalities.*

Stable theories are now well-understood. What can we say about unstable theories?

# A map of the (first-order) universe

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# Simple theories: Kim–Pillay ('90s and '00s)

For $k \in \omega$, a set of formulas $\{\varphi_n(x) \mid n \in \omega\}$ is $k$-inconsistent if any subset of size $k$ is inconsistent.

## Definition (Shelah '80)

$T$ is *simple* if no formula $\varphi(x; y)$ has the *tree property*: there exist tuples $(a_\eta)_{\eta \in \omega^{<\omega}}$ and $k \geq 2$ such that for all $\sigma \in \omega^\omega$, $\{\varphi(x; a_{\sigma|_n}) \mid n \in \omega\}$ is consistent, but for any $\eta \in \omega^{<\omega}$, $\{\varphi(x; a_{\eta\hat{}n}) \mid n \in \omega\}$ is $k$-inconsistent.

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Interest in simple theories increased when Kim and Pillay showed that forking independence, originally defined in stable theories, continues to behave well in the wider class of simple theories.

## Theorem (Kim '96)

*$T$ is simple if and only if $\mathop{\smile\!\!\!\!\!\!\mid}^f$ is symmetric: $A \mathop{\smile\!\!\!\!\!\!\mid}^f_C B \iff B \mathop{\smile\!\!\!\!\!\!\mid}^f_C A$.*

# Characterizing $\mathop{\smile\!\!\!\!\!\!|}^f$

## Theorem (Kim–Pillay '96)

*Let $T$ be a complete theory and $\mathop{\smile\!\!\!\!\!\!|}$ any relation on subsets of $\mathbb{M} \models T$. Then $T$ is simple and $\mathop{\smile\!\!\!\!\!\!|} = \mathop{\smile\!\!\!\!\!\!|}^f$ if and only if:*

- *Invariance: If $a \mathop{\smile\!\!\!\!\!\!|}_C b$ and $\mathrm{tp}(a'b'C') = \mathrm{tp}(abC)$, then $a' \mathop{\smile\!\!\!\!\!\!|}_{C'} b'$.*
- *Local character: For all $a$ and $B$, there is $C \subseteq B$ such that $|C| \leq |T|$ and $a \mathop{\smile\!\!\!\!\!\!|}_C B$.*
- *Finite character: $a \mathop{\smile\!\!\!\!\!\!|}_C B$ if and only if for every finite tuple $b$ from $B$, $a \mathop{\smile\!\!\!\!\!\!|}_C b$.*
- *Extension: For all $a$, $B$, and $C$, there exists $a'$ such that $\mathrm{tp}(a'/C) = \mathrm{tp}(a/C)$ and $a' \mathop{\smile\!\!\!\!\!\!|}_C B$.*
- *Symmetry: If $a \mathop{\smile\!\!\!\!\!\!|}_C b$, then $b \mathop{\smile\!\!\!\!\!\!|}_C a$.*
- *Transitivity: If $D \subseteq C \subseteq B$, then $a \mathop{\smile\!\!\!\!\!\!|}_D C$ and $a \mathop{\smile\!\!\!\!\!\!|}_C B$ if and only if $a \mathop{\smile\!\!\!\!\!\!|}_D B$.*
- *... and* **the independence theorem**: *see next slide.*

# The independence theorem

The most important condition in the axiomatic characterization of forking independence in simple theories is the *independence theorem*:

Let $M \models T$ be a model, $A$ and $B$ sets, and $a$ and $a'$ tuples such that $\mathrm{tp}(a/M) = \mathrm{tp}(a'/M)$. If $A \mathop{\smile\!\!\!\!\!\!|}_M B$, $a \mathop{\smile\!\!\!\!\!\!|}_M A$, and $a' \mathop{\smile\!\!\!\!\!\!|}_M B$, then there exists $a''$ such that

1. $\mathrm{tp}(a''A/M) = \mathrm{tp}(aA/M)$,
2. $\mathrm{tp}(a''B/M) = \mathrm{tp}(a'B/M)$, and
3. $a'' \mathop{\smile\!\!\!\!\!\!|}_M AB$.

# Generic theories are often simple

Examples of simple theories:

- ACFA: the generic theory of algebraically closed fields with an automorphism.
- The random graph (also the generic theories of bipartite graphs, directed graphs, and $k$-hypergraphs).
- The generic theory of all $L$-structures in a relational language $L$.

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**Theorem (Winkler '75)**

*Let $T$ be an $L$-theory which is model complete and eliminates $\exists^\infty$. Let $L'$ be a language containing $L$. Then $T$, considered as an $L'$-theory, has a model companion $T_{L'}$, the generic expansion of $T$ to $L'$.*

**Theorem (Chatzidakis–Pillay '98)**

*Generic relational expansions preserve simplicity: If $T$ is simple and the new symbols in $L'$ are all relation symbols, then $T_{L'}$ is simple.*

# The generic binary function

What happens with function symbols?

Let $T_L^\emptyset$ be the generic theory of all $L$-structures.
If $L$ contains a single binary function $f$, then already $T_L^\emptyset$ (the generic theory of magmas) is not simple.
The formula $\varphi(x; y_1, y_2)\colon f(x, y_1) = y_2$ has the tree property.

**Question (Jeřábek)**

Is $T_L^\emptyset$ always NSOP$_1$?

(Later, Jeřábek independently answered this question)

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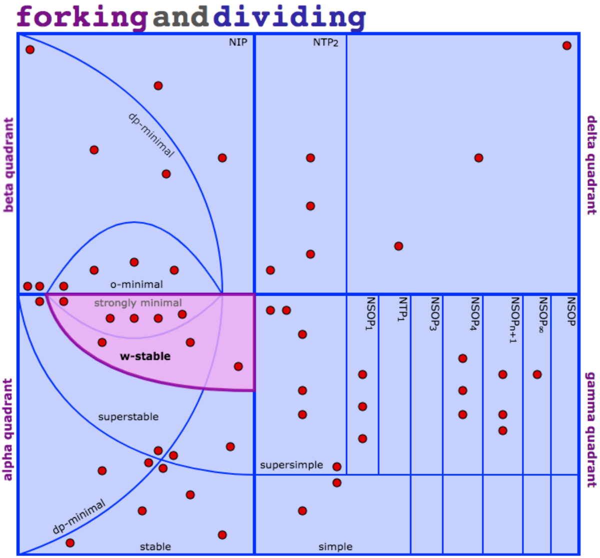


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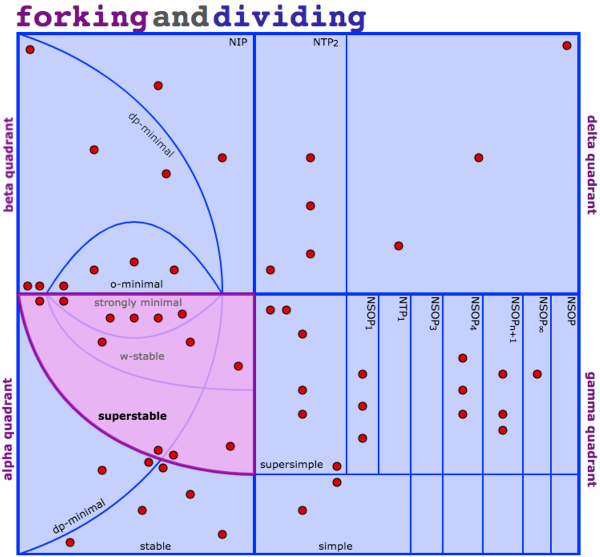


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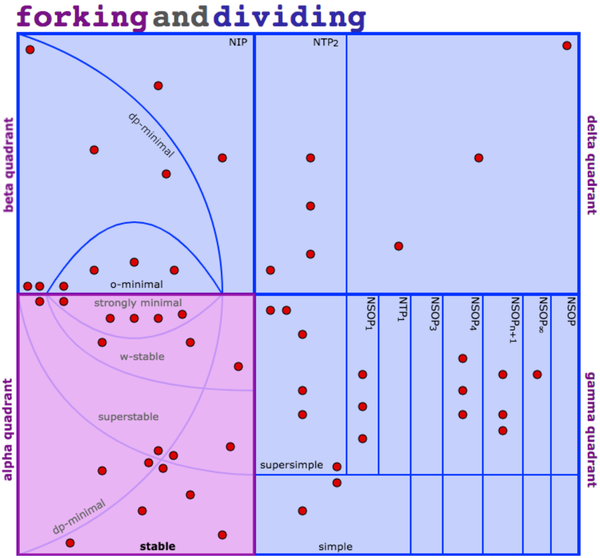


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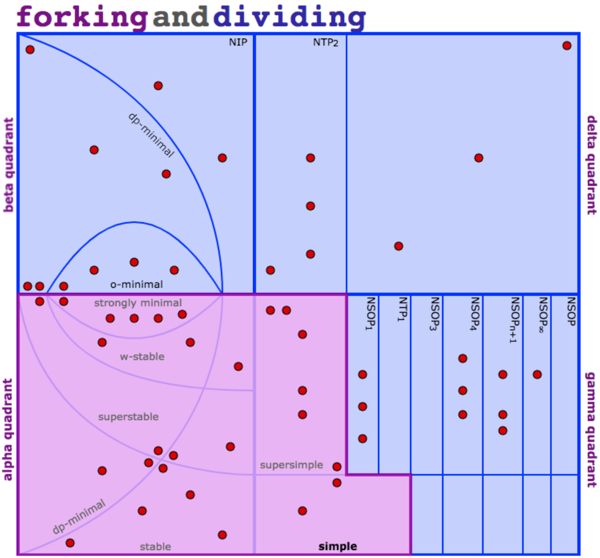


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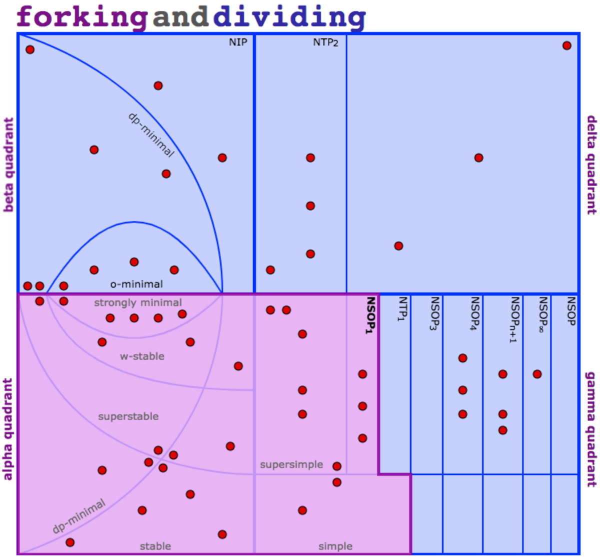


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# $\mathsf{NSOP}_1$ theories: Chernikov–Kaplan–Ramsey ('10s)

## Definition (Shelah '04)

$T$ is $\mathsf{NSOP}_1$ if no formula $\varphi(x;y)$ has $\mathsf{SOP}_1$: there exist tuples $(a_\eta)_{\eta\in 2^{<\omega}}$ such that for all $\sigma \in 2^\omega$, $\{\varphi(x; a_{\sigma|_n}) \mid n \in \omega\}$ is consistent, but for any $\nu, \eta \in 2^{<\omega}$, if $\nu\hat{}0 \leq \eta$, then $\{\varphi(x; a_\eta), \varphi(x; a_{\nu\hat{}1})\}$ is inconsistent.

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## Theorem (Kaplan–Ramsey '17)

$T$ is $\text{NSOP}_1$ if and only if $\mathop{\smile\hspace{-0.9em}\mid}^K$ is symmetric.

In any theory $T$, $\mathop{\smile\hspace{-0.9em}\mid}^f \implies \mathop{\smile\hspace{-0.9em}\mid}^K$. In simple theories, $\mathop{\smile\hspace{-0.9em}\mid}^f = \mathop{\smile\hspace{-0.9em}\mid}^K$.

# Characterizing $\mathop{\smile\hskip-0.9em\vert}^{K}$

Crucially, there is a Kim–Pillay style characterization of $\mathop{\smile\hskip-0.9em\vert}^{K}$ in $\text{NSOP}_1$.

**Theorem (Kaplan–Ramsey '17)**

*Let $T$ be a complete theory and $\mathop{\smile\hskip-0.9em\vert}$ any relation on subsets of $\mathbb{M} \models T$. Then $T$ is NSOP$_1$ and $\mathop{\smile\hskip-0.9em\vert}_M = \mathop{\smile\hskip-0.9em\vert}^{K}_M$ for all $M \models T$ if and only if:*

1. *Strong finite character and witnessing: if $a \not\mathop{\smile\hskip-0.9em\vert}_M b$, then there is a formula $\varphi(x, b, m) \in tp(a/bM)$ such that for any $a' \models \varphi(x, b, m)$, $a' \not\mathop{\smile\hskip-0.9em\vert}_M b$. Moreover, if $(b_i)_{i\in\omega}$ is a Morley sequence over $M$ in a global $M$-finitely satisfiable type extending $tp(b/M)$, then $\{\varphi(x, b_i, m) \mid i \in \omega\}$ is inconsistent.*
2. *Existence: $a \mathop{\smile\hskip-0.9em\vert}_M M$.*
3. *Monotonicity: if $aa' \mathop{\smile\hskip-0.9em\vert}_M bb'$, then $a \mathop{\smile\hskip-0.9em\vert}_M b$.*
4. *Symmetry: if $a \mathop{\smile\hskip-0.9em\vert}_M b$, then $b \mathop{\smile\hskip-0.9em\vert}_M a$.*
5. **The independence theorem.**

# Deficiencies of $\mathop{\smile\!\!\!\!\!\!|}\,^K$

The main property of $\mathop{\smile\!\!\!\!\!\!|}\,^f$ in simple theories which is lost by $\mathop{\smile\!\!\!\!\!\!|}\,^K$ in NSOP$_1$ theories is base monotonicity:

If $D \subseteq C \subseteq B$ and $A \mathop{\smile\!\!\!\!\!\!|}\,^f_D B$, then $A \mathop{\smile\!\!\!\!\!\!|}\,^f_C B$.

# Deficiencies of $\mathop{\smile\!\!\!\!\!\!|}^K$

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Also, we currently only know how to define $A \mathop{\smile\!\!\!\!\!\!|}^K_M B$ when $M \models T$.

It's an open problem to define $\mathop{\smile\!\!\!\!\!\!|}^K$ in a sensible way over arbitrary base sets.

# Generic $L$-structures

## Theorem (K.–Ramsey '17)

*Let $T_L^\emptyset$ be the generic theory of all $L$-structures.*

- $T_L^\emptyset$ *eliminates quantifiers, and* $\mathrm{acl}(A) = \langle A \rangle$, *the substructure generated by* $A$.
- $\mathop{\smile\!\!\!\!\!\!|}^a$ *satisfies the independence theorem over arbitrary sets.*
- *It follows easily that* $T_L^\emptyset$ *is NSOP*$_1$ *and* $\mathop{\smile\!\!\!\!\!\!|}^K = \mathop{\smile\!\!\!\!\!\!|}^a$ *over models.*
- $\mathop{\smile\!\!\!\!\!\!|}^f$ *is obtained by "forcing" base monotonicity on* $\mathop{\smile\!\!\!\!\!\!|}^a$*:* $A \mathop{\smile\!\!\!\!\!\!|}^f_C B$ *if and only if* $A \mathop{\smile\!\!\!\!\!\!|}^a_{C'} B$ *for all* $C \subseteq C' \subseteq \mathrm{acl}(BC)$.
  *e.g. generically, if* $f(a,b) = c$, *then* $bc \mathop{\smile\!\!\!\!\!\!|}^f_\emptyset a$, *but* $a \not\mathop{\smile\!\!\!\!\!\!|}^a_b c$, *so* $a \not\mathop{\smile\!\!\!\!\!\!|}^f_\emptyset bc$.
- *Forking = dividing for complete types, but when* $L$ *contains an* $n$*-ary function* ($n \geq 2$)*, there is a formula which forks but does not divide.*
- $T_L^\emptyset$ *has weak elimination of imaginaries.*

# Classification of the generic $L$-structure

| Relation arities: | $\leq 0$ | $\leq 1$ | any | any |
|---|---|---|---|---|
| Function arities: | $\leq 0$ | $\leq 1$ | $\leq 1$ | any |
| $T_L^\emptyset$ is: | uncountably categorical | stable* | simple* | $\text{NSOP}_1$ |

* If $T_L^\emptyset$ is stable/simple, then it is superstable/supersimple if and only if there is at most one unary function symbol in $L$.

# Generic Skolemization

Typically, a formula $\varphi(x; a)$ may have many solutions in $\mathbb{M}$, but there may be no *definable* solution. For example, the polynomial $x^2 = 2$ has two solutions in $\mathbb{C}$, which are conjugate by an automorphism.

A Skolem function $f_\varphi(y)$ for $\varphi(x; y)$ takes parameters as inputs and produces a solution to the formula when possible:

$$\forall y\, (\exists x\, \varphi(x; y) \to \varphi(f_\varphi(y); y)).$$

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$$\forall y\,(\exists x\,\varphi(x;y) \rightarrow \varphi(f_\varphi(y);y)).$$

**Theorem (Winkler '75)**

*Let $T$ be an $L$-theory which is model complete and eliminates $\exists^\infty$. Let $L_{\mathrm{Sk}} = L \cup \{f_\varphi(y) \mid \varphi(x;y) \text{ an } L\text{-formula}\}$. Let $T_+$ be $T$ together with the Skolem axiom given above for each $L$-formula $\varphi(x;y)$. Then $T_+$ has a model companion $T_{\mathrm{Sk}}$, the generic Skolemization of $T$.*

# Generic expansions and Skolemizations

**Theorem (K.–Ramsey '17)**

- *Generic expansions preserve NSOP$_1$: If $T$ is NSOP$_1$, then $T_{L'}$ is NSOP$_1$. Further, letting $\mathbb{M}' \models T_{L'}$ and $\mathbb{M} \models T$ be its reduct to $L$,*

$$a \mathop{\downarrow}^K_M b \text{ in } \mathbb{M}' \iff \operatorname{acl}_{L'}(Ma) \mathop{\downarrow}^K_M \operatorname{acl}_{L'}(Mb) \text{ in } \mathbb{M}.$$

- *Generic Skolemizations preserve NSOP$_1$: If $T$ is NSOP$_1$, then $T_{\mathrm{Sk}}$ is NSOP$_1$. Further, letting $\mathbb{M}' \models T_{\mathrm{Sk}}$ and $\mathbb{M} \models T$ be its reduct to $L$,*

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Again, the main difficulty is proving the independence theorem. The proof involves some technical work on the relationship between $\mathop{\downarrow}^K$ and $\mathop{\downarrow}^a$ in arbitrary NSOP$_1$ theories.

# Generic projective planes

An *incidence structure* is a structure in the language $\{P, L, I\}$, where:

- $P$ and $L$ are unary relation partitioning the structure into two disjoint sets ("points" and "lines")
- $I$ is a binary relation ("incidence") such that if $I(a, b)$ holds, then $a \in P$ and $b \in L$.

In other words, an incidence structure is a bipartite graph with the two halves of the partition named.

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An incidence structure $A$ is a *partial plane* if any two points are incident with *at most* one line and any two lines are incident with *at most* one point. Let $T^p_{2,2}$ be the theory of *partial* planes.

# Generic projective planes

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In other words, an incidence structure is a bipartite graph with the two halves of the partition named.

An incidence structure $A$ is a **projective plane** if any two points are incident with **exactly** one line and any two lines are incident with **exactly** one point. Let $T^c_{2,2}$ be the theory of **projective** planes.

# Generic projective planes

Equivalently, an incidence structure is a partial plane if it does not contain a copy of the complete bipartite graph $K_{2,2}$.

Almost everything I will say can be generalized to $T^p_{m,n}$ the theory of incidence structures which do not contain a copy of $K_{m,n}$, for $m, n \geq 2$. But we'll stick to projective planes for simplicity.

# Generic projective planes

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For any subset $A$ of a projective plane $B$, there is a smallest projective plane containing it, called its $I$-*closure*, obtained by iteratively adding the intersection points of all pairs of lines, and the connecting lines of all pairs of points.

**Theorem (Conant–K. '17)**

*$T^p_{2,2}$ has a model companion $T_{2,2}$, which is also the model companion of $T^c_{2,2}$. In $T_{2,2}$, $\mathrm{acl}$ coincides with $I$-closure, and $T_{2,2}$ eliminates quantifiers "down to existential quantifiers over the $I$-closure".*

# Independence and NSOP$_1$

Define $A \mathop{\smile\!\!\!\!\!\!\,|}^I_C B$ if and only if $A \mathop{\smile\!\!\!\!\!\!\,|}^a_C B$ and there are no incidences between $\mathrm{acl}(AC) \setminus \mathrm{acl}(C)$ and $\mathrm{acl}(BC) \setminus \mathrm{acl}(C)$.

We can then prove very similar results as in the case of $T_L^\emptyset$:

**Theorem (Conant–K. '17)**

- $\mathop{\smile\!\!\!\!\!\!\,|}^I$ *satisfies the independence theorem over arbitrary sets.*
- *It follows easily that $T_{2,2}$ is NSOP$_1$ and $\mathop{\smile\!\!\!\!\!\!\,|}^K = \mathop{\smile\!\!\!\!\!\!\,|}^I$ over models.*
- $\mathop{\smile\!\!\!\!\!\!\,|}^f$ *is obtained by "forcing" base monotonicity on $\mathop{\smile\!\!\!\!\!\!\,|}^I$: $A \mathop{\smile\!\!\!\!\!\!\,|}^f_C B$ if and only if $A \mathop{\smile\!\!\!\!\!\!\,|}^I_{C'} B$ for all $C \subseteq C' \subseteq \mathrm{acl}(BC)$.*
- *Forking $=$ dividing for complete types, but there is a formula which forks but does not divide.*
- *$T_{2,2}$ has weak elimination of imaginaries.*

# Failure of simplicity

Let $p_1$ and $p_2$ be points and $\ell_1$ and $\ell_2$ lines such that there are no incidences between the $p_i$ and $\ell_j$, but the unique line $\ell^*$ through $p_1$ and $p_2$ contains the unique point $p^*$ at the intersection of $\ell_1$ and $\ell_2$.

Then $p_1\ell_1 \mathop{\smile\!\!\!\!\!|}^{I}_{\emptyset} p_2\ell_2$. But $p^* \in \operatorname{acl}(p_1\ell_1\ell_2)$, so $\ell^* \in \operatorname{acl}(p_1\ell_1\ell_2)$, and $\ell^*$ is incident to $p_2$. Thus $p_1\ell_1 \not\mathop{\smile\!\!\!\!\!|}^{I}_{\ell_2} p_2\ell_2$, and $\mathop{\smile\!\!\!\!\!|}^{I}$ fails base monotonicity.

Letting $\varphi(x_1, x_2, x^*, y_1, y_2, y^*)$ be the conjunction of all atomic formulas satisfied by $(p_1, p_2, p^*, \ell_1, \ell_2, \ell^*)$, one can show that the formula $\psi(x_1, x_2; y_1, y_2)$: $\exists x^* \exists y^* \varphi(x_1, x_2, x^*, y_1, y_2, y^*)$ has the tree property.

# Countable models

A *quadrangle* is $4$ points in a projective plane, no $3$ of which lie on a line.

A projective plane is *desarguesian* (i.e. it satisfies the classical Desargues's Theorem) if and only if it is isomorphic to a projective plane over a division ring. In any desarguesian plane, any two quadrangles are conjugate by an automorphism.

In contrast, we explicitly construct continuum-many $4$-types over $\emptyset$ relative to $T_{2,2}$. This shows that $T_{2,2}$ has no countable saturated model.

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In contrast, we explicitly construct continuum-many $4$-types over $\emptyset$ relative to $T_{2,2}$. This shows that $T_{2,2}$ has no countable saturated model.

Every partial plane embeds in a projective plane. But the following is a longstanding open problem.

**Question (Erdős, essentially)**

Does every finite partial plane embed in a *finite* projective plane?

It turns out that $T_{2,2}$ has a countable prime model if and only if this question has a positive answer.