Presentating finitary functors

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In the paper *On Finitary Functors and their Presentations* [1], Adámek, Milius, and Moss prove that every finitary endofunctor on a locally finitely presentable category has a presentation by a signature and a set of flat equations. They give a rather abstract proof of this theorem, appealing in particular to Beck's Monadicity Theorem. In understanding this theorem, I found it useful to work out a more elementary proof, which I will present in this note.

We begin with the preliminaries and an explanatation of what the statement means. See [1] for more details and examples.

A functor F is *finitary* if it preserves filtered colimits. An object A in a category $\mathcal{C}$ is *finitely presentable* if the functor $\mathrm{Hom}(A, -)$ is finitary. The category $\mathcal{C}$ is *locally finitely presentable* if it is cocomplete and if there is a set $\mathcal{F}$ of finitely presentable objects such that every object in $\mathcal{C}$ is a filtered colimit of objects in $\mathcal{F}$. We will view $\mathcal{F}$ as a full subcategory of $\mathcal{C}$ and assume that it is skeletal (i.e. it contains exactly one representative from each isomorphism class). I will use lower case letters to denote objects in $\mathcal{F}$ and upper case letters to denote general objects of $\mathcal{C}$.

We will use the fact that every object A in $\mathcal{C}$ is the filtered colimit of a canonical diagram, consisting of all maps from objects in $\mathcal{F}$ to A. Formally, we let $I\colon \mathcal{F} \to \mathcal{C}$ be the inclusion and consider the (small) comma category $(I \downarrow A)$. An object of this category is a map $f\colon x \to A$ with $x \in \mathcal{F}$, and an arrow from $f\colon x \to A$ to $g\colon y \to A$ is a map $h\colon x \to y$ such that $f = g \circ h$. This category is filtered, and the functor $D\colon (I \downarrow A) \to \mathcal{C}$ given by $D(f\colon x \to A) = x$ is a diagram in $\mathcal{C}$ with colimit A. Of course, the co-cone map $D(f\colon x \to A) \to A$ is f itself. We will abbreviate this by writing

$$A = \varinjlim_{f\colon x \to A} x.$$

As a consequence, if F is a finitary functor, then

$$F(A) = \varinjlim_{f\colon x \to A} F(x),$$

and the co-cone map on the component indexed by f is $F(f)\colon F(x) \to F(A)$.

A *signature* Σ is collection $(\Sigma_x)_{x \in \mathcal{F}}$ of objects of $\mathcal{D}$, indexed by the objects in $\mathcal{F}$. To the signature Σ, we associate the *polynomial functor* $H_\Sigma\colon \mathcal{C} \to \mathcal{C}$, defined by

$$H_\Sigma(A) = \coprod_{f\colon x \to A} \Sigma_x.$$

If $\varphi\colon A \to B$ is an arrow, then $H_\Sigma(\varphi)\colon H_\Sigma(A) \to H_\Sigma(B)$ is the map from the coproduct defined on the $f\colon x \to A$ component by the inclusion of Σ_x in the $\varphi \circ f$ component of $H_\Sigma(B)$:

$$i_{\varphi\circ f}\colon \Sigma_x \to \coprod_{g\colon y \to B} \Sigma_y.$$

So we have $H_\Sigma(\varphi) \circ i_f = i_{\varphi\circ f}$.

A *flat equation* in the signature Σ is a parallel pair of maps $\rho, \rho' : x \to H_\Sigma(y)$. Given a set T of flat equations, let $T_{x,y}$ be the set of those pairs $(\rho, \rho') \in T$ with domain x and codomain $H_\Sigma(y)$. Then we define a new signature Σ_T by

$$(\Sigma_T)_y = \coprod_{(\rho,\rho')\in T_{x,y}} x.$$

Now we obtain a parallel pair of natural transformations $\eta, \eta'\colon H_{\Sigma_T} \to H_\Sigma$. The component η_A is a map $H_{\Sigma_T}(A) \to H_\Sigma(A)$:

$$\coprod_{f\colon y\to A} \coprod_{(\rho,\rho')\in T_{x,y}} x \to H_\Sigma(A)$$

which we define on the component of the coproduct indexed by f and (ρ, ρ') as the composition

$$x \xrightarrow{\rho} H_\Sigma(y) \xrightarrow{H_\Sigma(f)} H_\Sigma(A).$$

So we have $\eta_A \circ i_f \circ i_{(\rho,\rho')} = H_\Sigma(f) \circ \rho$. We define η' similarly, with ρ' in place of ρ.

Let's check naturality of η. Given a map $\varphi\colon A \to B$, we want to show that the following diagram commutes:

$$\begin{array}{ccc} H_{\Sigma_T}(A) & \xrightarrow{\eta_A} & H_\Sigma(A) \\ {\scriptstyle H_{\Sigma_T}(\varphi)}\downarrow & & \downarrow{\scriptstyle H_\Sigma(\varphi)} \\ H_{\Sigma_T}(B) & \xrightarrow[\eta_B]{} & H_\Sigma(B). \end{array}$$

Checking commutativity on the component of $H_{\Sigma_T}(A)$ indexed by $f\colon y \to A$ and $(\rho, \rho') \in T_{x,y}$,

$$\begin{aligned} H_\Sigma(\varphi) \circ \eta_A \circ i_f \circ i_{(\rho,\rho')} &= H_\Sigma(\varphi) \circ H_\Sigma(f) \circ \rho \\ &= H_\Sigma(\varphi \circ f) \circ \rho \\ &= \eta_B \circ i_{\varphi\circ f} \circ i_{(\rho,\rho')} \\ &= \eta_B \circ H_{\Sigma_T}(\varphi) \circ i_f \circ i_{(\rho,\rho')}. \end{aligned}$$

The exact sense in which the pair of natural transformations (η, η') encode the equations in T is captured by the following lemma.

Lemma. *Fix objects A and B and a map $\psi\colon H_\Sigma(A) \to B$. Then $\psi \circ \eta_A = \psi \circ \eta'_A$ if and only if for all flat equations $\rho, \rho'\colon x \to H_\Sigma(y)$ in T and all maps $f\colon y \to A$, we have $\psi \circ H_\Sigma(f) \circ \rho = \psi \circ H_\Sigma(f) \circ \rho'$.*

Proof. The equality $\psi \circ \eta_A = \psi \circ \eta'_A$ holds if and only if $\psi \circ \eta_A \circ i_f \circ i_{(\rho,\rho')} = \psi \circ \eta_A \circ i_f \circ i_{(\rho,\rho')}$ holds for all $f\colon y \to A$ and all $(\rho, \rho') \in T_{x,y}$. Reversing the quantifiers and rewriting, this is true if and only if for all $\rho, \rho'\colon x \to H_\Sigma(y)$ in T and all maps $f\colon y \to A$, we have $\psi \circ H_\Sigma(f) \circ \rho = \psi \circ H_\Sigma(f) \circ \rho'$. □

Let $G\colon \mathcal{C} \to \mathcal{C}$ be a functor, and let $\varepsilon\colon H_\Sigma \to G$ be a natural transformation. We say that G is *presented by the equations T in the signature* Σ if ε is the coequalizer of η and η' in the category of functors $\mathcal{C} \to \mathcal{C}$:

$$H_{\Sigma_T} \underset{\eta'}{\overset{\eta}{\rightrightarrows}} H_\Sigma \overset{\varepsilon}{\to} G.$$

It is a fact that all polynomial functors and all functors presented by flat equations are finitary. Our goal is to prove that the converse is true, and we now have enough definitions in place to do so (the reader may skip to the proof of the Theorem below). But first, I will provide some motivation for this notion, observing that if a functor F is presented by the equations T in signature Σ, then an F-algebra is "the same as" an H_Σ-algebra which satisfies all the equations in T.

Let $F\colon \mathcal{C} \to \mathcal{C}$ be a functor. An *F-algebra* is an object $A \in \mathcal{C}$ together with a map $\alpha\colon F(A) \to A$.

Let (A, α) be an H_Σ-algebra. We say that A *satisfies* the flat equation $\rho, \rho'\colon x \to H_\Sigma(y)$ if for all maps $f\colon y \to A$, we have $\alpha \circ H_\Sigma(f) \circ \rho = \alpha \circ H_\Sigma(f) \circ \rho'$. Taking $B = A$ and $\psi = \alpha$ in the Lemma, we see that (A, α) satisfies all the equations in T if and only if $\alpha \circ \eta_A = \alpha \circ \eta'_A$

If $\varepsilon\colon F \to G$ is a natural transformation, then any G-algebra (A, α) also carries the structure of an F-algebra, with structure map $\alpha \circ \varepsilon_A\colon F(A) \to A$.

Claim. *If G is presented by the equations T in the signature Σ, then the above correspondence puts G-algebras in bijection with H_Σ-algebras which satisfy all the equations in T.*

Proof. Suppose (A, α) is a G-algebra. We claim that the H_Σ-algebra $(A, \alpha \circ \varepsilon_A)$ satisfies all the equations in T. As we observed above, it suffices to show that $\alpha \circ \varepsilon_A \circ \eta_A = \alpha \circ \varepsilon_A \circ \eta'_A$. But this is true, since ε is the coequalizer of η and η'.

Conversely, suppose (A, α) is an H_Σ-algebra which satisfies all the equations in T. Then $\alpha \circ \eta_A = \alpha \circ \eta'_A$. Since ε_A is the coequalizer of η_A and η'_A, we obtain an arrow $\beta\colon G(A) \to A$, which makes A into a G-algebra:

$$\begin{array}{ccccc} H_{\Sigma_T}(A) & \underset{\eta'_A}{\overset{\eta_A}{\rightrightarrows}} & H_\Sigma(A) & \xrightarrow{\varepsilon_A} & G(A) \\ & & & \searrow^{\alpha} & \downarrow{\scriptstyle \beta} \\ & & & & A. \end{array}$$

And $\beta \circ \varepsilon_A = \alpha$, so these correspondences are inverses. □

With a little more work, one can show that this bijection gives an isomorphism of categories between the category G-Alg and the full subcategory of H_Σ-Alg consisting of those algebras satisfying all the equations in T.

Now let's prove the main theorem.

Theorem. *Every finitary functor $F\colon \mathcal{C} \to \mathcal{C}$ can be presented by a signature and a set of flat equations.*

Proof. First, we define the signature Σ. For all $x \in \mathcal{F}$, let $\Sigma_x = F(x)$. Thus

$$H_\Sigma(A) = \coprod_{f\colon x \to A} F(x).$$

There is a natural transformation $\varepsilon\colon H_\Sigma \to F$. Its component $\varepsilon_A\colon H_\Sigma(A) \to F(A)$ is defined on the component of the coproduct indexed by $f\colon x \to A$ as $F(f)\colon F(x) \to F(A)$.

As an aside, this is a very natural choice of signature. Since F is finitary, for any object A, ε_A is just the obvious map

$$\coprod_{f\colon x \to A} F(x) \to \varinjlim_{f\colon x \to A} F(x) = F(A)$$

which is the identity $\mathrm{id}_{F(x)}$ on the component indexed by $f\colon x \to A$, followed by the co-cone map $F(f)$.

Let's check that ε is natural. Given a map $\varphi\colon A \to B$, we want to show that the following diagram commutes:

$$\begin{array}{ccc} H_\Sigma(A) & \xrightarrow{\varepsilon_A} & F(A) \\ {\scriptstyle H_\Sigma(\varphi)}\downarrow & & \downarrow{\scriptstyle F(\varphi)} \\ H_\Sigma(B) & \xrightarrow{\varepsilon_B} & F(B). \end{array}$$

Checking commutativity on the component of $H_\Sigma(A)$ indexed by $f\colon x \to A$,

$$\begin{aligned} F(\varphi) \circ \varepsilon_A \circ i_f &= F(\varphi) \circ F(f) \\ &= F(\varphi \circ f) \\ &= \varepsilon_B \circ i_{\varphi \circ f} \\ &= \varepsilon_B \circ H_\Sigma(\varphi) \circ i_f. \end{aligned}$$

Let $T = \{\rho, \rho'\colon x \to H_\Sigma(y) \mid \varepsilon_y \circ \rho = \varepsilon_y \circ \rho'\}$. As above, the set T of flat equations induces natural transformations $\eta, \eta'\colon H_{\Sigma_T} \to H_\Sigma$. Our claim is that ε is the coequalizer of η and η'.

First, we show that $\varepsilon \circ \eta = \varepsilon \circ \eta'$. Fix an object A, and consider the component of $H_{\Sigma_T}(A)$ indexed by $f\colon y \to A$ and $(\rho, \rho') \in T_{x,y}$. Since $\varepsilon_A \circ \eta_A \circ i_f \circ i_{(\rho,\rho')} = \varepsilon_A \circ H_\Sigma(f) \circ \rho$ and similarly for η'_A, we would like to show that

$\varepsilon_A \circ H_\Sigma(f) \circ \rho = \varepsilon_A \circ H_\Sigma(f) \circ \rho'$. This follows from naturality of ε, and the fact that $\varepsilon_y \circ \rho = \varepsilon_y \circ \rho'$:

$$\begin{array}{ccccc} x & \overset{\rho}{\underset{\rho'}{\rightrightarrows}} & H_\Sigma(y) & \xrightarrow{H_\Sigma(f)} & H_\Sigma(A) \\ & & \downarrow{\scriptstyle \varepsilon_y} & & \downarrow{\scriptstyle \varepsilon_A} \\ & & F(y) & \xrightarrow[F(f)]{} & F(A). \end{array}$$

Next, suppose there is another functor G and a natural transformation $\varepsilon^* \colon H_\Sigma \to G$ such that $\varepsilon^* \circ \eta = \varepsilon^* \circ \eta'$. We must show that there is a unique natural transformation $\zeta \colon F \to G$ making the diagram commute:

$$\begin{array}{ccccc} H_{\Sigma_T} & \overset{\eta}{\underset{\eta'}{\rightrightarrows}} & H_\Sigma & \xrightarrow{\varepsilon} & F \\ & & & \searrow^{\varepsilon^*} & \downarrow{\scriptstyle \zeta} \\ & & & & G. \end{array}$$

Fix an object A. To define $\zeta_A \colon F(A) \to G(A)$, it suffices to give a coherent family of maps $\zeta_A^f \colon F(x) \to G(A)$, since

$$F(A) = \varinjlim_{f \colon x \to A} F(x).$$

Then the triangle that we want to commute can be rewritten as

$$\begin{array}{ccc} \coprod_{f \colon x \to A} F(x) & \xrightarrow{\varepsilon_A} & \varinjlim_{f \colon x \to A} F(x) \\ & \searrow^{\varepsilon_A^*} & \downarrow{\scriptstyle \zeta_A} \\ & & G(A) \end{array}$$

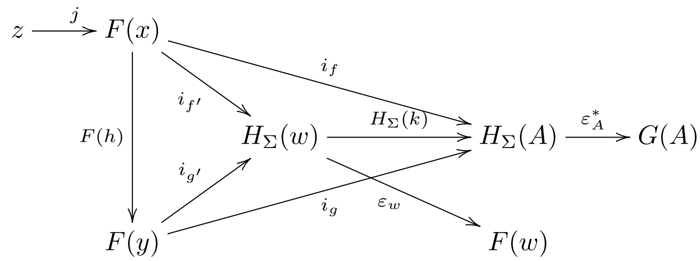
Recalling that ε_A maps the component of the coproduct indexed by f to the component of the colimit indexed by f by the identity, we are forced to define $\zeta_A^f = \varepsilon_A^* \circ i_f$ (where, as usual, i_f is the inclusion of $F(x)$ in the component of the coproduct indexed by f). The fact that this choice was forced on us establishes uniqueness.

It remains to check that the maps ζ_A^f cohere to a map $F(A) \to G(A)$. That is, given a commutative triangle

$$\begin{array}{ccc} x & \xrightarrow{f} & A \\ \downarrow{\scriptstyle h} & \nearrow_{g} & \\ y & & \end{array} \qquad \begin{array}{ccc} F(x) & \xrightarrow{\zeta_A^f} & G(A) \\ \downarrow{\scriptstyle F(h)} & \nearrow_{\zeta_A^g} & \\ F(y) & & \end{array}$$

we must show that $\zeta_A^f = \zeta_A^g \circ F(h)$. Equivalently, that $\varepsilon_A^* \circ i_f = \varepsilon_A^* \circ i_g \circ F(h)$.

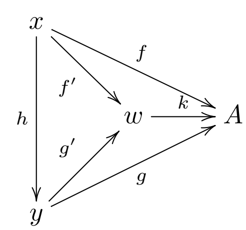
For reference, here is a diagram containing all of the arrows which will be used in the proof:



We now use the fact that $\mathcal{C}$ is locally finitely presentable in two key ways.

Since $F(x)$ is a filtered colimit of finitely presentable objects, it suffices to show that for all $j\colon z \to F(x)$, $\varepsilon_A^* \circ i_f \circ j = \varepsilon_A^* \circ i_g \circ F(h) \circ j$.

Since x and y are finitely presentable and A is a filtered colimit of finitely presentable objects, the maps $f\colon x \to A$ and $g\colon y \to A$ factor through some map $k\colon w \to A$ as in the following diagram:



Now consider the flat equation $\rho, \rho'\colon z \to H_\Sigma(w)$, where we define $\rho = i_{f'} \circ j$ and $\rho' = i_{g'} \circ F(h) \circ j$. I claim that $(\rho, \rho') \in T$, i.e. that $\varepsilon_w \circ \rho = \varepsilon_w \circ \rho'$. Indeed,

$$\begin{aligned}
\varepsilon_w \circ \rho' &= \varepsilon_w \circ i_{g'} \circ F(h) \circ j \\
&= F(g') \circ F(h) \circ j \\
&= F(f') \circ j \\
&= \varepsilon_w \circ i_{f'} \circ j \\
&= \varepsilon_w \circ \rho.
\end{aligned}$$

Now by the Lemma, applied to the flat equation $\rho, \rho'\colon z \to H_\Sigma(w)$ and the map $k\colon w \to A$, since $\varepsilon_A^* \circ \eta_A = \varepsilon_A^* \circ \eta'_A$, we have $\varepsilon_A^* \circ H_\Sigma(k) \circ \rho = \varepsilon_A^* \circ H_\Sigma(k) \circ \rho'$. Rewriting,

$$\begin{aligned}
\varepsilon_A^* \circ H_\Sigma(k) \circ i_{f'} \circ j &= \varepsilon_A^* \circ H_\Sigma(k) \circ i_{g'} \circ F(h) \circ j \\
\varepsilon_A^* \circ i_{k \circ f'} \circ j &= \varepsilon_A^* \circ i_{k \circ g'} \circ F(h) \circ j \\
\varepsilon_A^* \circ i_f \circ j &= \varepsilon_A^* \circ i_g \circ F(h) \circ j
\end{aligned}$$

as was to be shown. □

References

[1] Adámek J., Milius S., Moss L.S. (2012) On Finitary Functors and Their Presentations. In: Pattinson D., Schrder L. (eds) Coalgebraic Methods in Computer Science. CMCS 2012. Lecture Notes in Computer Science, vol 7399. Springer, Berlin, Heidelberg.