Foundations of Cologic

Alex Kruckman

Indiana University, Bloomington

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- Corelations and costructures
- Syntax and semantics
- Motivation 2: The cologic of profinite groups
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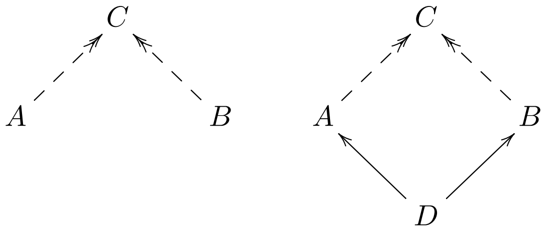

(arrows are embeddings)

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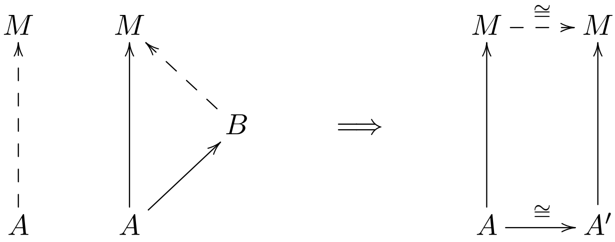

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- The first-order theories of Fraïssé limits are exactly the countably categorical theories with quantifier elimination. The "embedding extension properties" are expressible by $\forall\exists$ axioms.

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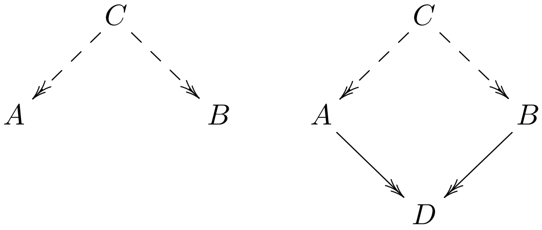
K is a projective Fraïssé class if it is closed under quotients by coembeddings (dual to HP), and satisfies the duals of JEP and AP.



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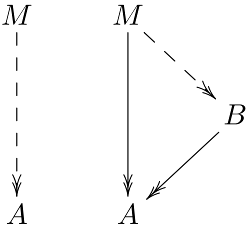
Under these hypotheses, K has a projective Fraïssé limit M, constructed as an inverse limit of finite structures in K along coembeddings, and satisfying the duals of universality and homogeneity.



Here the maps out of M are *continuous* coembeddings:

- As an inverse limit of finite sets, M is naturally a Stone space (compact, Hausdorff, zero-dimensional = basis of clopen sets).
- We equip the finite structures in K with the discrete topology.

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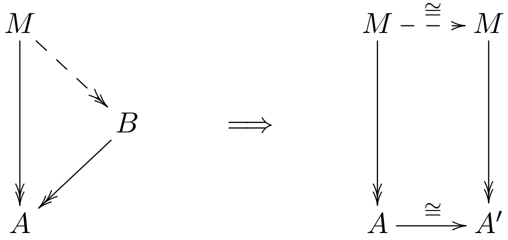

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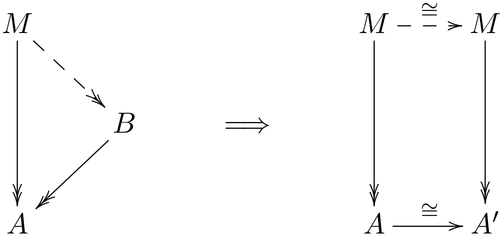

Idea: There should be a full first-order "cologic", in which:

- Finite quotients play the role of finite substructures.
- The "coembedding extension properties" are expressible as $\forall\exists$ axioms.
- Projective Fraïssé theory is the special case of "cocountably categorical" cotheories with quantifier elimination.

Corelations and costructures

For ordinary first-order structures:

An n-tuple from M is a map $[n] \to M$ ($[n] = \{1, \ldots, n\}$).

We denote the set of n-tuples by M^n.

An n-ary relation R on M is a set of n-tuples from M, $R \subseteq M^n$.

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Definition (Solecki)

An n-*cotuple* from a Stone space M is a continuous map $M \to [n]$.
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Definition

A *costructure* for the corelational cosignature $\mathcal{R}$ is a Stone space M together with an $\mathrm{ar}(R)$-ary corelation R^M on M for each $R \in \mathcal{R}$.

Corelations and costructures

We sometimes think of an n-cotuple $f\colon M \to [n]$ as a labeled partition of M into n clopen sets, $f^{-1}(\{1\}), \ldots, f^{-1}(\{n\})$.

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Corelations (and hence coformulas) express *properties of partitions*.

Coembeddings

Let M and N be costructures. Given a continous map $f\colon M \to N$, any n-cotuple $A\colon N \to [n]$ pulls back to an n-cotuple $A \circ f\colon M \to [n]$.

$$M \xrightarrow{f} N \xrightarrow{A} [n]$$

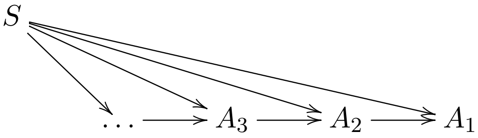
Definition

A continuous map $f\colon M \to N$ is a *coembedding* if

1. It is surjective, and
2. $A \in R^N$ if and only if $(A \circ f) \in R^M$, for every n-ary corelation R in the language and every n-cotuple A from N.

Why topology?

Set ≅ ind-FinSet: Every set is the filtered colimit of its finite subsets.
Stone ≅ pro-FinSet: Every Stone space is the cofiltered limit of its discrete finite quotients.



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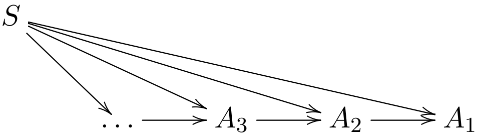

The topology on S captures the pro-structure:

- A basic clopen set in S is the preimage of a subset of A_i for some i.
- A map $T \to S$ is continuous iff it is induced by a coherent family of maps between the finite quotients of T and S.

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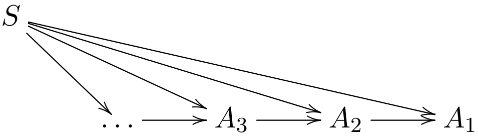

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Slogan: Logic explores infinite structures via their finite subsets.
Cologic explores infinite costructures via their finite quotients.

A final word on projective Fraïssé theory

It appears that corelational signatures provide the the "correct" general context for projective Fraïssé theory.

Theorem (Panagiotopoulos, '16)

1. *Let S be any second-countable Stone space. Then a subgroup G of $\mathrm{Homeo}(S)$ is closed in the compact-open topology if and only if $G = \mathrm{Aut}(M)$ for some projective Fraïssé limit M with domain S in a corelational signature.*
2. *Let Y be any second-countable compact Hausdorff space. Then there is a projective Fraïssé class K in a corelational signature, such that the projective Fraïssé limit M admits a canonical equivalence relation $\sim$, and $M/\sim$ is homeomorphic to Y.*

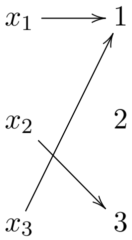
This generalizes many previous examples in which compact Hausdorff spaces were realized as quotients of projective Fraïssé limits.

Syntax

Let $X = \{x_1, \ldots, x_k\}$ be a finite set of "covariables". An n-cotuple of covariables in context X is a map $t\colon X \to [n]$.

We can represent an n-cotuple by an n-tuple describing a partition of X.

Example: $(x_1 \sqcup x_3, \emptyset, x_2)$ is a 3-cotuple in context $X = \{x_1, x_2, x_3\}$.



Think of the covariables as labeling a clopen partition of a costructure M.

Syntax

An atomic coformula in context X is:

- $R(t(X))$, where t is an $\mathrm{ar}(R)$-cotuple of covariables in context X.
- $\boxtimes_i t(X)$, where t is an n-cotuple of covariables in context X and $1 \leq i \leq n$. (This is the dual of equality.)

A coformula in context X is:

- An atomic coformula in context X.
- A Boolean combination of coformulas in context X.
- $\exists(y \sqcup z = x_i)\, \psi(\{x_1, \ldots, y, z, \ldots, x_n\})$, or $\forall(y \sqcup z = x_i)\, \psi(\{x_1, \ldots, y, z, \ldots, x_n\})$, where ψ is a coformula in context $(X \setminus \{x_i\}) \cup \{y, z\}$

Semantics

Let M be a costructure given together with an X-cotuple $A\colon M \to X$, and let $\varphi(X)$ be a coformula in variable context X. We will define the satisfaction relation $M \models \varphi(A)$.

Any n-cotuple of covariables in context X, $t\colon X \to [n]$, induces an n-cotuple $t \circ A\colon M \to [n]$ from M by composition, which we denote $t(A)$.

$$M \xrightarrow[A]{} X \xrightarrow[t]{} [n], \qquad M \xrightarrow{t(A)} [n]$$

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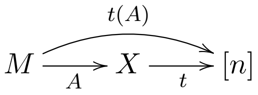
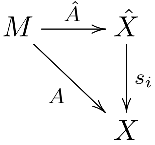


- $M \models R(t(A))$ iff $t(A) \in R^M$.
- $M \models \boxtimes_i\, t(A)$ iff $t(A)^{-1}(\{i\}) = \emptyset$.
- The usual satisfaction rules hold for Boolean combinations.

Note: In ordinary logic, equality tests for injectivity of a tuple $[n] \to M$. In cologic, coequality ($\boxtimes_i$) tests for surjectivity of a cotuple $M \to [n]$.

Semantics for quantifiers

Let $X = \{x_1, \dots, x_n\}$, and let $\hat{X} = (X \setminus \{x_i\}) \cup \{y, z\}$.
Define $s_i \colon \hat{X} \to X$ by $s_i(x_j) = x_j$ for $j \neq i$ and $s_i(y) = s_i(z) = x_i$.
A *lift* of the X-cotuple $A \colon M \to X$ is an $\hat{X}$-cotuple $\hat{A} \colon M \to \hat{X}$ such that $s_i \circ \hat{A} = A$.



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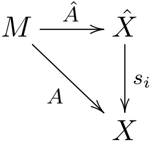

- $M \models \exists(y \sqcup z = x_i)\, \psi(\{x_1, \ldots, y, z, \ldots, x_n\})$ iff *there exists* a lift $\hat{A} \colon M \to \hat{X}$, such that $M \models \psi(\hat{A})$.
- $M \models \forall(y \sqcup z = x_i)\, \psi(\{x_1, \ldots, y, z, \ldots, x_n\})$ iff *for all* lifts $\hat{A} \colon M \to \hat{X}$, $M \models \psi(\hat{A})$.

Note: A lift $\hat{A}$ of A is a refinement of the partition given by A. So we are quantifying over finer partitions of M.

Cosentences and cotheories

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Theorem

Stone spaces are coelementarily equivalent (i.e. they satisfy the same cosentences in the empty language) if and only if their Boolean algebras of clopen sets are elementarily equivalent.

Elementary classes of Boolean algebras are classified by Tarski invariants.

Motivation 2: The cologic of profinite groups

[Cherlin, van den Dries, Macintyre, Chatzidakis]

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CDM cologic is just ordinary first-order logic on a multisorted structure encoding the full inverse system of finite quotients of a profinite group $\mathcal{G}$:

- One sort for each $n \geq 1$. Sort n consists of the disjoint union of all finite quotients of $\mathcal{G}$ of size n.
- A ternary relation $\cdot_n$ for each sort n, such that $\cdot_n(x, y, z)$ iff all three elements live in the same finite quotient of size n, and $x \cdot y = z$.
- A binary relation $\pi_{m,n}$ for each pair of sorts $m \geq n$, such that $\pi_{m,n}(x, y)$ iff $x \in H_1$ of size m, $y \in H_2$ of size n, and the quotient map $\pi_{H_2} \colon \mathcal{G} \to H_2$ factors through the quotient map $\pi_{H_1} \colon \mathcal{G} \to H_1$, as $\pi_{H_2} = \rho \circ \pi_{H_1}$, and $\rho(x) = y$.

Profinite groups as costructures

Option 1: Let L be the corelational signature with one n-ary corelation symbol R_H for each finite group H with domain $[n]$.

To any profinite group $\mathcal{G}$, we can associate a canonical L-costructure with the same underlying Stone space G:
Let $A\colon G \to [n]$ be an n-cotuple. Then, for any group H with domain $[n]$, $G \models R_H(A)$ if and only if $A\colon \mathcal{G} \to H$ is a surjective group homomorphism.

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Option 2: Develop cologic for pro-C objects, where C is any category with finite limits.

- Taking $C = \text{FinSet}$ gives cologic in the sense of this talk.
- Taking $C = \text{FinGrp}$ makes profinite groups costructures in the empty language. e.g. For a finite group H, an "H-cotuple" from a profinite group $\mathcal{G}$ is a continuous homomorphism $\mathcal{G} \to H$.

This can be done! In fact, it seems like the right level of generality.

Costructures as presheaf structures

Following CDM, we can encode a costructure M as an ordinary many-sorted first-order structure:

- One sort S_n for each $n \geq 0$, interpreted as $[n]^M$ $(= \mathrm{Hom}(M, [n]))$.
- One function symbol $f\colon S_m \to S_n$ for each function $f\colon [m] \to [n]$, interpreted as $(f \circ -)\colon [m]^M \to [n]^M$.
- One unary relation symbol R on sort S_n for each corelation symbol R, interpreted in the obvious way.

A structure in this language is called a *presheaf structure*, because the sorts and functions encode a functor FinSet $\to$ Set (a presheaf on FinSet$^{\mathrm{op}}$).

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$\mathsf{Stone}^{\mathrm{op}} = (\text{pro-}\mathsf{FinSet})^{\mathrm{op}}$ can be identified via $S \mapsto \mathrm{Hom}(S, -)$ with the subcategory of $\mathsf{Set}^{\mathsf{FinSet}}$ consisting of those functors FinSet $\to$ Set which preserve finite limits. This gives rise to a duality:

$$(\text{Costructures}, \text{Coembeddings})^{\mathrm{op}} \cong (\mathrm{Mod}(T_{\lim}), \text{Embeddings})$$

Where $T_{\lim}$ is the theory in the language of presheaf structures asserting that finite limits are preserved.

Costructures as presheaf structures

The sentences and formulas of cologic correspond to sentences and formulas in a *fragment* of first-order logic over presheaf structures: all formulas are unary, all quantifiers are bounded $[\forall(x \in f^{-1}(y))\, \varphi(x)]$, etc. But modulo $T_{\lim}$, this fragment is essentially as expressive as full first-order logic.

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Having "interpreted" cologic in many-sorted first-order logic, we get:

Corollary (Compactness)

A cotheory T is satisfiable if and only if every finite subset of T is satisfiable.

Costructures as presheaf structures

The sentences and formulas of cologic correspond to sentences and formulas in a *fragment* of first-order logic over presheaf structures: all formulas are unary, all quantifiers are bounded $[\forall(x \in f^{-1}(y))\, \varphi(x)]$, etc. But modulo $T_{\lim}$, this fragment is essentially as expressive as full first-order logic.

Having "interpreted" cologic in many-sorted first-order logic, we get:

Corollary (Compactness)

A cotheory T is satisfiable if and only if every finite subset of T is satisfiable.

Duality is useful for transporting theorems from ordinary first-order logic to cologic. But I also believe that it's valuable to have a natural syntax for cologic that refers directly to the intended semantics, rather than resorting to duality at every turn.

Motivation 3: Ultracoproducts of compact spaces

[Bankston]

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Ultraproducts can be described categorically as direct limits of products:

$$\prod_U M_i = \prod_{i \in I} M_i / U \cong \varinjlim_{X \in U} \prod_{i \in X} M_i$$

with connecting maps the projections $\prod_{i \in X} M_i \to \prod_{i \in Y} M_i$ when $Y \subseteq X$.

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To define the ultracoproduct, just dualize:

$$\coprod_U M_i \cong \varprojlim_{X\in U} \coprod_{i\in X} M_i$$

with connecting maps the inclusions $\coprod_{i\in Y} M_i \to \coprod_{i\in X} M_i$ when $Y \subseteq X$.

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Problem: When the M_i are sets, $\coprod_{i \in X} M_i$ is disjoint union, $\varprojlim$ is intersection, and when U is nonprincipal, $\coprod_U M_i$ is empty!

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Bankston's Solution: Work in the category of compact Hausdorff spaces. Then the infinite coproduct $\coprod_{i \in I} M_i$ is the Stone-Čech compactification of the disjoint union, and the intersection is nontrivial.

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Bankston has developed a remarkable amount of dualized model theory for compact Hausdorff spaces, without any syntax.

For example, two spaces S and T are defined to be colementarily equivalent if and only if they have homeomorphic ultracopowers!

Ultracoproducts of costructures

Let $(M_i)_{i\in I}$ be a family of costructures, and let U be an ultrafilter on I.
Define $\coprod_U M_i = \varprojlim_{X\in U} \coprod_{i\in X} M_i$ (limits and coproducts taken in Stone).

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The discrete space $[n]$ is *cocompact* in Stone: the functor $\mathrm{Hom}(-,[n])$ turns cofiltered limits into filtered colimits.

$$\begin{aligned}\mathrm{Hom}(\varprojlim_{X\in U} \coprod_{i\in X} M_i, [n]) &\cong \varinjlim_{X\in U} \mathrm{Hom}(\coprod_{i\in X} M_i, [n]) \\ &\cong \varinjlim_{X\in U} \prod_{i\in X} \mathrm{Hom}(M_i, [n])\end{aligned}$$

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So an n-cotuple $A\colon \coprod_U M_i \to [n]$ is determined by an n-cotuple $A_i\colon M_i \to [n]$ for each i, modulo equality on a set in the ultrafilter.

Define $\coprod_U M_i \models R(A)$ iff $\{i\in I \mid M_i \models R(A_i)\} \in U$.

Łos's theorem

Let $PS(M)$ be the presheaf structure corresponding to the costructure M.

We have $PS(\coprod_U M_i) \cong \prod_U PS(M_i)$.

Using duality, or adapting the usual proof:

Theorem (Łos's theorem for cologic)

Let U be an ultrafilter on I, and let $\{M_i \mid i \in I\}$ be a family of costructures. Let $\varphi(X)$ be a coformula in context X, let $A\colon \coprod_U M_i \to X$ be an X-cotuple from M, and let $(A_i\colon M_i \to X)_{i \in I}$ be any lift of A to a family of X-cotuples from the M_i. Then $\coprod_U M_i \models \varphi(A)$ if and only if $\{i \in I \mid M_i \models \varphi(A_i)\} \in U$.

Łos's theorem also gives a direct proof of the compactness theorem for cologic, in the usual way.

Motivation 4: Coalgebraic logic

[Rutten, Adámek, Kurz, Rosický, Moss, etc.]

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Let $\mathcal{C}$ be a category, and let F be a functor $\mathcal{C} \to \mathcal{C}$.

An F-algebra is an object A and a map $\eta\colon F(A) \to A$.

Example: $F\colon \text{Set} \to \text{Set}$, $F(X) = X^2 \sqcup C$ (C a set).
An F-algebra is a set A and a map $\eta\colon A^2 \sqcup C \to A$, determined by maps $f\colon A^2 \to A$ and $c\colon C \to A$, i.e. a structure for the signature with one binary function symbol f and a set C of constant symbols.

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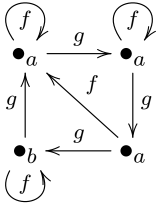
An F-coalgebra is an object A and a map $\varepsilon\colon A \to F(A)$.

Example: $F\colon \text{Stone} \to \text{Stone}$, $F(X) = X^2 \times C$ (C a Stone space).
An F-coalgebra is a Stone space S and a map $\eta\colon S \to S^2 \times C$, determined by maps $f\colon S \to S$, $g\colon S \to S$, and $c\colon S \to C$.

Motivation 4: Coalgebraic logic

The *constant space* C could be any Stone space, e.g. the underlying space of a costructure. For this example, we'll take $C = \{a, b\}$.

A coalgebra for the functor $F(X) = X^2 \times C$ is an transition system with inputs f and g, labeled by the constants C.



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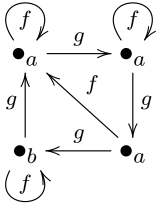

There are notions of "universal coalgebra" and "varieties of coalgebras" for coalgebras on Set, dual to classical universal algebra. But the "coalgebraic logics" in these frameworks are infinitary. Cologic gives a finitary compact logic for coalgebras on Stone.

Coterms

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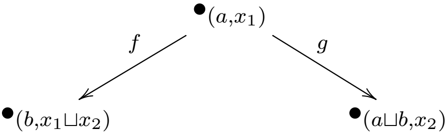
Answer: *The cofree coalgebra on the covariables.*

The cofree coalgebra on the covariables $X = \{x_1, x_2\}$ (for example) contains elements witnessing all possible behaviors under $\{f, g\}$-transitions and labelling by C and X (i.e. complete binary trees labeled by $C \times X$). A clopen set from the cofree coalgebra corresponds to a finite partial description of such a behavior:



Coterms

For example, let $C(X)$ be the cofree coalgebra on X. There is a 2-coterm (a continuous map $t\colon C(X) \to [2]$) described by

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Then the sentence $\forall(x_1 \sqcup x_2 = *) \boxtimes_1 (\bullet_{(x_1,a)} \xrightarrow{f} \bullet_{(x_2,b)}, \text{everything else})$ asserts that no clopen partition can separate two points s and t, such that s is labeled by a, t is labeled by b, and $f(s) = t$.

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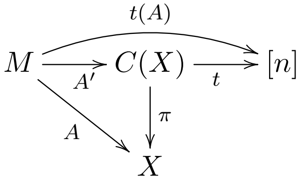
It's not hard to give a concrete syntax for coterms for simple functors like F, but it's work in progress to extend this to a more class of *cofinitary functors* on Stone.

Syntax and semantics, this time with coalgebra

Let $C(X)$ be the cofree coalgebra on X. An n-coterm in context X is an n-cotuple from $C(X)$, a continuous map $t\colon C(X) \to [n]$.
An atomic coformula in context X is:

- $R(t(X))$, where t is an $\mathrm{ar}(R)$-coterm in context X.
- $\boxtimes_i\, t(X)$, where t is an n-coterm in context X and $1 \leq i \leq n$.

Let M be a costructure given together with an X-cotuple $A\colon M \to X$. A induces a canonical map $A'\colon M \to C(X)$, and any n-coterm in context X, $t\colon C(X) \to [n]$, induces an n-cotuple $t \circ A'$, which we denote $t(A)$.



- $M \models R(t(A))$ if and only if $t(A) \in R^M$.
- $M \models \boxtimes_i\, t(A)$ if and only if $t(A)^{-1}(\{i\}) = \emptyset$.

Future plans

- Cologic on general pro-categories (& logic on general ind-categories!).
- Syntax for terms for general cofinitary functors.
- Develop more model theory, e.g. stability theory.
- Consider more general compact Hausdorff costructures (as suggested by the work of Panagiotopoulos and Bankston).
- Explore possible connections to:
  - Applications of coalgebras, e.g. in modal logic
  - Dual Ramsey theory (via "coindiscernibles"?)
- Try to combine ordinary logic and cologic into a compact second-order logic, via an $\in_{i,j}$ relation between tuples $a\colon [m] \to M$ and cotuples $A\colon M \to [n]$:

$$M \models a \in_{i,j} A \text{ iff } A(a(i)) = j \text{ (i.e. } a_i \in A_j).$$

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