First-order logic for locally finitely presentable categories and their duals

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Motivation: Cologic

Pieces of first-order logic and model theory have been successfully dualized:

- (Cherlin, van den Dries, and Macintyre; Chatzidakis) The first-order "cologic" of profinite groups (e.g. absolute Galois groups of fields).
- (Irwin and Solecki; Panagiotopoulos; others) Projective Fraïssé theory.
- (Rutten; Moss; others) Coalgebraic logic and universal coalgebra.
- (Bankston) Ultracoproducts and coelementary classes of compact Hausdorff spaces.

Question: Is there a unified framework encompassing all these examples?

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Answer: Yes.

First, we need to define a general categorical setting for first-order logic that's easy to dualize.
Given time, we'll return to these examples very briefly at the end.

Syntax

$\mathcal{C}$ is a small category, called the category of *contexts*.

Define the logic $\mathrm{FO}_{\mathcal{C}}$ inductively. For every object $x \in \mathcal{C}$, a formula in context x is

- $\top_x$ or $\bot_x$.
- $(\psi \wedge \theta)$, $(\psi \vee \theta)$, or $\neg\psi$, where ψ and θ are formulas in context x.
- $\exists_f \psi$, where $f : x \to y$ is an arrow in $\mathcal{C}$ and ψ is a formula in context y.

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Now suppose $\mathcal{C}$ is a subcategory of $\mathcal{D}$, called the category of *domains*.

If x is a context in $\mathcal{C}$ and M is a domain in $\mathcal{D}$, an arrow $a\colon x \to M$ is called an *interpretation* of x in M.

Semantics

We give a semantics in $\mathcal{D}$ for the logic $\mathrm{FO}_{\mathcal{C}}$ by defining the relation $M \models \varphi(a)$ inductively. For every domain $M \in \mathcal{D}$, every formula φ in context x, and every interpretation $a\colon x \to M$,

- If φ is $\top_x$, then $M \models \varphi(a)$. If φ is $\bot_x$, then $M \not\models \varphi(a)$.
- If φ is $(\psi \wedge \theta)$, then $M \models \varphi(a)$ iff $M \models \psi(a)$ and $M \models \theta(a)$.
- If φ is $(\psi \vee \theta)$, then $M \models \varphi(a)$ iff $M \models \psi(a)$ or $M \models \theta(a)$.
- If φ is $\neg\psi$, then $M \models \varphi(a)$ iff $M \not\models \psi(a)$.
- If φ is $\exists_f \psi$, for $f\colon x \to y$, then $M \models \varphi(a)$ iff there exists $b\colon y \to M$ such that $bf = a$ and $M \models \psi(b)$.

$$\begin{tikzcd} y \arrow[r, dashed, "\exists b"] & M \\ x \arrow[u, "f"] \arrow[ur, "a"'] & \end{tikzcd}$$

Example: L-structures

Fix a first-order signature L.

$\mathcal{D}$, the category of L-structures (and L-homomorphisms).
$\mathcal{C}$, the full category of finitely presentable L-structures.

Theorem

$\mathrm{FO}_{\mathcal{C}}$, *with semantics in* $\mathcal{D}$, *has essentially the same expressive power as first-order logic on* L*-structures.*

Translation: First-order to $\mathrm{FO}_{\mathcal{C}}$

We translate a first-order formula φ with free variables from a finite set X to an $\mathrm{FO}_{\mathcal{C}}$ formula in context $T(X)$, the term algebra on X.

- If φ is atomic, let $\widehat{\varphi}$ be $\exists_q \top_{\langle X \mid \{\varphi\}\rangle}$, where $q\colon T(X) \to \langle X \mid \{\varphi\}\rangle$ is the obvious map.
- If φ is $\psi \wedge \theta$, $\psi \vee \theta$, or $\neg\psi$, let $\widehat{\varphi}$ be $\widehat{\psi} \wedge \widehat{\theta}$, $\widehat{\psi} \vee \widehat{\theta}$, or $\neg\widehat{\psi}$, respectively.
- If φ is $\exists x'\, \psi$, where ψ is a formula with free variables from $X \cup \{x'\}$, let $\widehat{\varphi}$ be $\exists_i \widehat{\psi}$, $i\colon T(X) \to T(X')$ is the obvious map.

$$\begin{tikzcd} \langle X \mid \{\varphi\}\rangle \arrow[r, dashed, "\exists b"] & M \\ T(X) \arrow[u, "q"] \arrow[ur, "a"'] & \end{tikzcd} \qquad \begin{tikzcd} T(X') \arrow[r, dashed, "\exists b"] & M \\ T(X) \arrow[u, "i"] \arrow[ur, "a"'] & \end{tikzcd}$$

Translation: $\mathrm{FO}_{\mathcal{C}}$ to first-order

We translate an $\mathrm{FO}_{\mathcal{C}}$ formula in context x to a first-order formula with free variables from X, a finite set of generators of x.

- If φ is $\top_x$, let $\widetilde{\varphi}$ be $\top$. If φ is $\bot_x$, let $\widetilde{\varphi}$ be $\bot$.
- If φ is $\psi \wedge \theta$, $\psi \vee \theta$, or $\neg\psi$, let $\widetilde{\varphi}$ be $\widetilde{\psi} \wedge \widetilde{\theta}$, $\widetilde{\psi} \vee \widetilde{\theta}$, or $\neg\widetilde{\psi}$, respectively.
- If φ is $\exists_f \psi$, where $f \colon x \to y$ and ψ is a formula in context y:
    - Pick a finite presentation $\langle \{y_1, \ldots, y_\ell\} \mid \{\delta_1, \ldots, \delta_m\} \rangle$ for y.
    - For each $x_j \in X$, pick a term t_j in Y such that $t_j(\overline{y}) = f(x_j)$.
    - Let $\widetilde{\varphi}$ be

$$\exists y_1 \ldots \exists y_n \left(\left(\bigwedge_{i=1}^{m} \delta_i(\overline{y}) \right) \wedge \left(\bigwedge_{j=1}^{n} x_j = t_j(\overline{y}) \right) \wedge \widetilde{\psi}(\overline{y}) \right).$$

$$\begin{tikzcd} y \arrow[r, dashed, "\exists b"] & M \\ x \arrow[u, "f"] \arrow[ur, "a"'] & \end{tikzcd}$$

Locally finitely presentable categories

Question: Returning to general $\mathcal{C}$ and $\mathcal{D}$, what properties do we need to get a well-behaved logic $\mathrm{FO}_{\mathcal{C}}$?

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For the rest of this talk, let's assume:

- $\mathcal{C}$ has finite colimits.
- The objects of $\mathcal{D}$ are the directed colimits along diagrams in $\mathcal{C}$.
- Every object $x \in \mathcal{C}$ is *finitely presentable* in the sense that $\mathrm{Hom}_{\mathcal{D}}(x, -)$ preserves directed colimits (every map $x \to \varinjlim y_i$ factors through some y_i).

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In other words,

1. $\mathcal{D}$ is a *locally finitely presentable category*, and $\mathcal{C}$ is equivalent to its full subcategory of finitely presentable objects.
2. $\mathcal{D}$ is equivalent to $\mathrm{ind}-\mathcal{C}$, the formal co-completion of $\mathcal{C}$ under directed colimits.

Locally finitely presentable categories

Definition (Gabriel & Ulmer)

A category $\mathcal{D}$ is *locally finitely presentable* (LFP) if:

- It is co-complete.
- Every object is a directed colimit of finitely presentable objects.
- The full subcategory $\mathcal{F}$ of finitely presentable objects is essentially small, i.e. there is a set of isomorphism representatives of $\mathcal{F}$.

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Examples:

- Set; Set^X, for any set X; $\mathcal{D}^{\mathcal{B}}$, for any LFP $\mathcal{D}$ and small category $\mathcal{B}$.
- Str_L; Grp; Ring; Poset; Cat; Mod_T, where T is a first-order universal Horn theory.
- $\mathrm{Lex}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set})$, the finite-limit preserving presheaves on $\mathcal{C}$, for any small category $\mathcal{C}$ with finite colimits.
- The duals of ProFinSet $\cong$ Stone $\cong$ $\mathsf{Bool}^{\mathrm{op}}$ and ProFinGrp.

Presheaf structures

Gabriel-Ulmer duality tells us that for any LFP category $\mathcal{D}$,

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with the equivalence given by $M \mapsto \mathrm{Hom}_{\mathcal{D}}(-, M)$.

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Let L_{PSh} be the ordinary first-order language consisting of:

- A sort S_x for each $x \in \mathcal{C}$.
- A function symbol $\widetilde{f}$ of sort $S_y \to S_x$ for each arrow $f\colon x \to y$.

Let T_{PSh} be the first-order theory asserting:

- $x \mapsto S_x$ is a functor $\mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$ (i.e. $\widetilde{f \circ g} = \widetilde{g} \circ \widetilde{f}$ and $\widetilde{\mathrm{id}} = \mathrm{id}$).
- This functor preserves limits.

The first-order translation

Theorem

$\mathrm{FO}_{\mathcal{C}}$, *with semantics in* $\mathcal{D}$, *has essentially the same expressive power as first-order logic in the language* L_{PSh} *on models of* T_{PSh}.

This first-order translation implies we can import theorems (compactness, Löwenheim-Skolem, etc.) and definitions (stability, NIP, etc.) from first-order model theory for free.

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...But doesn't it make $\mathrm{FO}_{\mathcal{C}}$ redundant? I don't think so:

1. $\mathrm{FO}_{\mathcal{C}}$ seems more natural than the many-sorted L_{PSh} in examples.
2. Understanding this case will be useful in generalizing beyond LFP categories, where we may not have a first-order translation.

Sequents and sentences

I'll describe a sequent calculus proof system for $\mathrm{FO}_{\mathcal{C}}$, modeled on the notion of a hyperdoctrine.

A *sequent* has the form $\varphi \Rightarrow_x \psi$, where φ and ψ are formulas in context x.

A domain M satisfies $\varphi \Rightarrow_x \psi$ if $M \models \varphi(a)$ implies $M \models \psi(a)$ for every interpretation $a\colon x \to M$.

A *sentence* is a formula in context 0 (the initial object). Every sequent $\varphi \Rightarrow_x \psi$ is equivalent to the sentence $\forall_!(\varphi \to \psi)$, where $!\colon 0 \to x$ is the unique arrow.

Hyperdoctrines

Definition

Let $\mathcal{B}$ be a category with finite limits. A *first-order (Boolean) hyperdoctrine* over $\mathcal{B}$ is a functor $P\colon \mathcal{B}^{\mathrm{op}} \to \mathsf{Bool}$, such that for every arrow $f\colon y \to x$ in $\mathcal{B}$, the Boolean homomorphism $Pf\colon Px \to Py$ has a left adjoint, i.e. a monotone map $\exists_f\colon Py \to Px$ such that

$$\varphi \leq_{Py} Pf(\psi) \iff \exists_f \varphi \leq_{Px} \psi,$$

satisfying the Beck-Chevalley condition: For every pullback square in $\mathcal{B}$,

$$\begin{tikzcd} w \arrow[r, "f'"] \arrow[d, "g'"'] & z \arrow[d, "g"] \\ y \arrow[r, "f"'] & x \end{tikzcd}$$

and every $\varphi \in Py$, we have $Pg(\exists_f(\varphi)) = \exists_{f'}(Pg'(\varphi))$.

Substitution

We need a new formula-building operation to play the role of Pf in the hyperdoctrine.

$[\varphi]_f$ is a formula in context y, when φ is a formula in context x and $f\colon x \to y$ is an arrow in $\mathcal{C}$.

Semantics: Given a domain M and an interpretation $b\colon y \to M$, $M \models [\varphi]_f(b)$ iff $M \models \varphi(bf)$.

$$\begin{tikzcd}
y \arrow[r, "b"] & M \\
x \arrow[u, "f"] \arrow[ur, dashed, "bf"'] &
\end{tikzcd}$$

It will follow from our proof rules that every formula is equivalent to one built without any instances of substitution.

Propositional rules

$$\frac{}{\varphi \Rightarrow_x \varphi}\ \text{REF} \qquad \frac{\varphi \Rightarrow_x \psi \quad \psi \Rightarrow_x \theta}{\varphi \Rightarrow_x \theta}\ \text{TRANS} \qquad \frac{}{\varphi \Rightarrow_x \top_x}\ \text{TRUE}$$

$$\frac{\varphi \Rightarrow_x \psi \quad \varphi \Rightarrow_x \theta}{\varphi \Rightarrow_x \psi \wedge \theta}\ \text{AND} \qquad \frac{}{\psi \wedge \theta \Rightarrow_x \psi}\ \text{AND}_L \qquad \frac{}{\psi \wedge \theta \Rightarrow_x \theta}\ \text{AND}_R$$

$$\frac{\psi \Rightarrow_x \varphi \quad \theta \Rightarrow_x \varphi}{\psi \vee \theta \Rightarrow_x \varphi}\ \text{OR} \qquad \frac{}{\psi \Rightarrow_x \psi \vee \theta}\ \text{OR}_L \qquad \frac{}{\theta \Rightarrow_x \psi \vee \theta}\ \text{OR}_R$$

$$\frac{}{\varphi \wedge (\psi \vee \theta) \Rightarrow_x (\varphi \wedge \psi) \vee (\varphi \wedge \theta)}\ \text{DIST} \qquad \frac{}{\bot_x \Rightarrow_x \varphi}\ \text{FALSE}$$

$$\frac{}{\top_x \Rightarrow_x \varphi \vee \neg\varphi}\ \text{NOT}_1 \qquad \frac{}{\varphi \wedge \neg\varphi \Rightarrow_x \bot_x}\ \text{NOT}_2$$

Substitution rules

For all arrows $f\colon x \to y$ and $g\colon y \to z$ in $\mathcal{C}$,

$$\frac{}{\varphi \Leftrightarrow_x [\varphi]_{\mathrm{id}_x}}\ \mathrm{ID} \qquad \frac{}{[\varphi]_{gf} \Leftrightarrow_z [[\varphi]_f]_g}\ \mathrm{COMP} \qquad \frac{\varphi \Rightarrow_x \psi}{[\varphi]_f \Rightarrow_y [\psi]_f}\ \mathrm{MON}$$

$$\frac{}{\top_y \Rightarrow_y [\top_x]_f}\ \mathrm{HOM}_\top \qquad \frac{}{[\bot_x]_f \Rightarrow_y \bot_y}\ \mathrm{HOM}_\bot$$

$$\frac{}{[\psi]_f \wedge [\theta]_f \Rightarrow_y [\psi \wedge \theta]_f}\ \mathrm{HOM}_\wedge \qquad \frac{}{[\psi \vee \theta]_f \Rightarrow_y [\psi]_f \vee [\theta]_f}\ \mathrm{HOM}_\vee$$

Quantifier rules

For every arrow $f\colon x \to y$ in $\mathcal{C}$,

$$\frac{\varphi \Rightarrow_y \psi}{\exists_f\varphi \Rightarrow_x \exists_f\psi}\ \mathrm{MON}_\exists \qquad \frac{}{\varphi \Rightarrow_y [\exists_f\varphi]_f}\ \mathrm{UNIT} \qquad \frac{}{\exists_f[\theta]_f \Rightarrow_x \theta}\ \mathrm{COUNIT}$$

For every pushout square,

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "g"'] & y \arrow[d, "g'"] \\ z \arrow[r, "f'"'] & w \end{tikzcd} \qquad \frac{}{[\exists_f\varphi]_g \Rightarrow_z \exists_{f'}[\varphi]_{g'}}\ \mathrm{BC}$$

Completeness and compactness

It's easy to check that these rules are sound.

Theorem (Completeness)

Let T be a set of sequents. Then $T \models \varphi \Rightarrow_x \psi$ if and only if $T \vdash \varphi \Rightarrow_x \psi$.

Proof idea: If $T \nvdash \varphi \Rightarrow_x \psi$, build a countermodel to $\varphi \Rightarrow_x \psi$ as the colimit of a directed system from $\mathcal{C}$, carefully adding witnesses to the necessary existential quantifiers.

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Categorical interpretation: The logic $\mathrm{FO}_{\mathcal{C}}$ is the initial hyperdoctrine over $\mathcal{C}^{\mathrm{op}}$, and it has a natural semantics in ind-$\mathcal{C}$.

Presenting LFP categories by signatures

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In contrast, every map between finite sets can be decomposed into a composition of two kinds of maps: adding a new point and identifying two points.

Traditional first-order logic takes $\mathcal{C} = \mathsf{FinSet}$ (in which the arrows, and hence quantifiers, are easy to understand) and adds extra structure via a signature and atomic formulas.

Signatures and structures

Fix categories $\mathcal{C}$ and $\mathcal{D}$ as before.

Definition

A signature Σ consists of, for every context $x \in \mathcal{C}$,

- *A set $\mathfrak{R}_x$, called the x-ary relation symbols.*
- *A finitary (commutes with directed colimits) endofunctor $F\colon \mathcal{D} \to \mathcal{D}$.*

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Definition

A Σ-structure is a domain M in $\mathcal{D}$, together with, for every context $x \in \mathcal{C}$,

- *An "x-ary relation" $R^M \subseteq \mathrm{Hom}_{\mathcal{D}}(x, M)$ for each $R \in \mathfrak{R}_x$.*
- *An F-algebra structure on M, i.e. a map $\eta\colon F(M) \to M$.*

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Adámek, Milius, and Moss showed that finitary functors on LFP categories can be presented as quotients of "signature functors" by "flat equations". This allows for a definition of signatures in terms of "function symbol" objects, and a more concrete description of terms (omitted here).

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Definition

Let $x, y \in \mathcal{C}$. A y-term in context x is an arrow $y \to T(x)$.

Given a y-term t in context x and an interpretation $a\colon x \to M$, we obtain a map $t^M(a)$, the "evaluation of t in M".

$$\begin{tikzcd}
y \arrow[r, "t"'] \arrow[rr, bend left, "t^M(a)"] & T(x) \arrow[r, dashed] & M \\
& x \arrow[u, "i"'] \arrow[ur, "a"'] &
\end{tikzcd}$$

Atomic formulas

Given a signature Σ, we build a logic $\mathrm{FO}_{\mathcal{C}}(\Sigma)$ as before, but with new atomic formulas.

Definition

Let $x \in \mathcal{C}$ be a context. An atomic formula in context x is one of the following:

- $s(x) = t(x)$, *where s and t are y-terms in context x, for some $y \in \mathcal{C}$.*
- $R(t(x))$, *where t is a y-term in context x and R is a y-ary relation symbol, for some $y \in \mathcal{C}$.*

Given a Σ-structure M, a formula $\varphi(x)$ in context x, and $a \colon x \to M$:

- $M \models s(a) = t(a)$ iff $s^M(a) = t^M(a)$ in $\mathrm{Hom}_{\mathcal{D}}(y, M)$.
- $M \models R(t(a))$ iff $t^M(a) \in R^M \subseteq \mathrm{Hom}_{\mathcal{D}}(y, M)$.

Cologic

Now we can dualize: If $\mathcal{D}^{\mathrm{op}}$ is LFP (e.g. if $\mathcal{D} = \text{pro-}\mathcal{C}$), we can form the logic $\mathrm{FO}_{\mathcal{C}^{\mathrm{op}}}(\Sigma)$ with semantics in $\mathcal{D}^{\mathrm{op}}$.

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"Corelations" and "coformulas" express properties of "cotuples" $M \to x$. Σ-structures are now *coalgebras* for *cofinitary* functors on $\mathcal{D}$.

Example: $\mathcal{D} = \text{Stone}$, $\mathcal{C} = \text{FinSet}$ (viewed as finite discrete spaces). An interpretation $a\colon M \to x$ is a partition of M into $|x|$ clopen sets.
$\exists_f$ quantifies over refinements of this partition and tests for emptiness of pieces of the partition (coequality) when f is not surjective.

$$\begin{tikzcd}
M \arrow[r, dashed, "\exists b"] \arrow[dr, "a"'] & y \arrow[d, "f"] \\
& x
\end{tikzcd}$$

Cologic - unifying the examples

- The cologic of profinite groups, as defined by Cherlin, van den Dries, and Macintyre, takes place is a multi-sorted first-order setting, which is is essentially the same as the first-order translation (via presheaf structures) of $\mathrm{FO}_{\mathsf{FinGrp}^{\mathrm{op}}}$.
- In the case of Stone spaces, Bankston's coelementary classes are exactly $\mathrm{FO}_{\mathsf{FinSet}^{\mathrm{op}}}$-elementary classes.
- Panagiotopoulos showed that any projective Fraïssé limit can be viewed as the limit of a class of finite structures in a corelational signature Σ. The $\mathrm{FO}_{\mathsf{FinSet}^{\mathrm{op}}}(\Sigma)$-theories of projective Fraïssé limits are characterized by "$\aleph_0$-categoricity" and quantifier elimination.
- Coalgebras for cofinitary functors on Stone are of some interest. For example, coalgebras for the Vietoris functor are exactly the descriptive general frames in modal logic. "Universal coalgebra" in this setting is captured by "equational theories" in $\mathrm{FO}_{\mathsf{FinSet}^{\mathrm{op}}}(\Sigma)$.

Future Work

1. Generalize to categories which are not locally finitely presentable. In particular, it would be interesting to extend the framework to include:
    1. Coalgebras on Set (possibly via Stone-Čech compactification).
    2. Compact Hausdorff spaces (inspired by Bankston's work on coelementary classes in this category) and compact groups.
2. In concrete profinite structures, both the tuples and cotuples are interesting. Is there a nice logic which talks about both at once?
3. Study model theoretic properties: nontrivial $\mathrm{FO}_{\mathsf{FinSet}^{\mathrm{op}}}(\Sigma)$-theories *always* have the strict order property and the independence property (these are bad), but $\mathrm{FO}_{\mathsf{FinGrp}^{\mathrm{op}}}(\Sigma)$-theories can be model-theoretically tame (since they are interpretable in reasonable theories of fields). What's the deeper reason for this?

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