Sampling Measures and Limits of Finite Structures

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Definition (from combinatorics)

A sequence of finite structures $\langle A_i \rangle_{i \in \omega}$ *converges* if the sequence of real numbers $\langle p(\phi; A_i) \rangle_{i \in \omega}$ converges for all quantifier-free formulas ϕ.

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What does the sequence $\langle A_i \rangle_{i \in \omega}$ converge to?

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$$\left([\exists x\, \phi(\overline{a}, x)] = \bigcup_{b \in \omega} [\phi(\overline{a}, b)]\right)$$

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Fact

Let μ be an invariant measure. The following are equivalent:

1. *For any quantifier-free formulas ϕ and ψ and disjoint tuples $\overline{x}$ and $\overline{y}$, $\mu(\phi(\overline{x}) \wedge \psi(\overline{y})) = \mu(\phi(\overline{x}))\mu(\psi(\overline{y}))$.*
2. *The same, for ϕ and ψ formulas of $L_{\omega_1,\omega}$.*
3. *μ is ergodic, i.e. for every almost surely invariant Borel set $B \subseteq X_\Sigma$, $\mu(B) = 0$ or 1.*

We will call such a measure a sampling measure.

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For every sentence $\phi \in L_{\omega_1,\omega}$, $[\phi]$ is invariant, so $\mu(\phi) = 0$ or 1.

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Theorem (Lovász)

Every sampling measure is induced by some convergent sequence of finite structures.

Two questions

Theorem (Scott)

For any countable structure M, there is a sentence ϕ_M of $L_{\omega_1,\omega}$ such that $N \models \phi_M$ if and only if $N \cong M$.

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$T_\mu = \{\phi$ a sentence of $L_{\omega_1,\omega} \mid \mu(\phi) = 1\}$

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Answer: No!

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Definition

A *fragment* F of $L_{\omega_1,\omega}$ is a set of formulas which is closed under subformula, finite Boolean combinations, quantification, and exchange of free variables (from a fixed countable supply).

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If F is countable, and p is an F-type, then $\Phi_p(\overline{x}) = \bigwedge_{\phi \in p} \phi(\overline{x})$ is a formula of $L_{\omega_1,\omega}$. Write $\mu(p)$ as shorthand for $\mu(\Phi_p)$.

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For a given sampling measure μ, build a sequence of countable fragments of $L_{\omega_1,\omega}$, $\{F_\alpha\}_{\alpha\in\omega_1}$:

$$F_0 = \text{the first-order fragment.}$$

$$F_\gamma = \text{the fragment generated by } \bigcup_{\alpha<\gamma} F_\alpha, \text{ for } \gamma \text{ a limit ordinal.}$$

$$F_{\alpha+1} = \text{the fragment generated by } F_\alpha \cup \{\Phi_p \mid p \in S^n(F_\alpha, \mu), n \in \omega\}.$$

Stabilization

Say $p \in S^n(F_\alpha, \mu)$ *splits later* if for some $\beta > \alpha$, every type $q \in S^n(F_\beta, \mu)$ with $p \subset q$ has $\mu(q) < \mu(p)$.

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Fix $n \in \omega$.
For each $\alpha \in \omega_1$, let

$$\begin{aligned} \mathsf{Sp}(\alpha) &= \{p \in S^n(F_\alpha, \mu) \mid p \text{ splits later}\}, \\ r_\alpha &= \sup\{\mu(p) \mid p \in \mathsf{Sp}(\alpha)\}. \end{aligned}$$

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So there is some stage $\beta > \alpha$ such that each all the p_i have split before β. All types in $\mathsf{Sp}(\beta)$ have measure strictly less than r_α^n, so $r_\beta^n < r_\alpha^n$.

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So there is some stage $\beta > \alpha$ such that each all the p_i have split before β. All types in $\mathsf{Sp}(\beta)$ have measure strictly less than r^n_α, so $r^n_\beta < r^n_\alpha$.

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Hence there is some $\gamma_n \in \omega_1$ such that $\mathsf{Sp}(\gamma_n)$ is empty: No type in $S^n(F_{\gamma_n}, \mu)$ splits later. Let $\gamma = \sup_{n<\omega}\{\gamma_n\} \in \omega_1$.

Concentration

Proposition

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For $q \in S^{n+1}(F_\gamma, \mu)$, the formula $\exists y \Phi_q(\overline{x}, y)$ is in $F_{\gamma+1}$, and

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For each p, μ gives measure 1 to the sentence

$$\forall \overline{x} \left(\Phi_p(\overline{x}) \rightarrow \left(\forall y \bigvee_{\substack{q \in S^{n+1}(F_\gamma,\mu) \\ p \subset q}} \Phi_q(\overline{x}, y) \wedge \bigwedge_{\substack{q \in S^{n+1}(F_\gamma,\mu) \\ p \subset q}} \exists y\, \Phi_q(\overline{x}, y) \right)\right)$$

Finish by a back-and-forth argument.

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Proof.

If μ does not concentrate on a countable structure, then for some n,

$$\sum_{p \in S^n(F_\gamma, \mu)} \mu(p) < 1.$$

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Then $\mu(\bigwedge_{p \in S^n(F_\gamma,\mu)} \neg \Phi_p(\overline{x})) > 0$, so $\exists \overline{x} \bigwedge_{p \in S^n(F_\gamma,\mu)} \neg \Phi_p(\overline{x}) \in T_\mu$.

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So if T_μ has a model M, some $\overline{a}$ from M has F_γ-type $q \notin S^n(F_\alpha, \mu)$. But $\mu(q) = 0$, so $\mu(\exists \overline{x}\, \Phi_q(\overline{x})) = 0$, and $\forall \overline{x}\, \neg\Phi_q(\overline{x}) \in T_\mu$, contradiction. □

Vaught's conjecture for sampling measures

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If μ is a sampling measure and ϕ is a sentence of $L_{\omega_1,\omega}$ with fewer than continuum many countable models such that $\mu(\phi) = 1$, then μ concentrates on a model of ϕ.

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- $S^n(F, \phi)$ is the set of F-types in n free variables consistent with ϕ.
- For any countable fragment F, and any n and ϕ, $S^n(F, \phi)$ is an analytic ($\mathbf{\Sigma}^1_1$) set in 2^F, so if $|S^n(F, \phi)| < 2^{\aleph_0}$, then $|S^n(F, \phi)|$ is countable.

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- For any countable fragment F, and any n and ϕ, $S^n(F, \phi)$ is an analytic ($\mathbf{\Sigma_1^1}$) set in 2^F, so if $|S^n(F, \phi)| < 2^{\aleph_0}$, then $|S^n(F, \phi)|$ is countable.
- Since a countable structure realizes at most countably many F-types, and our ϕ has fewer than continuum many models, $|S^n(F, \phi)|$ is countable for all countable fragments F and all n.

Vaught's conjecture for sampling measures

Corollary

If μ is a sampling measure and ϕ is a sentence of $L_{\omega_1,\omega}$ with fewer than continuum many countable models such that $\mu(\phi) = 1$, then μ concentrates on a model of ϕ.

Proof.

For any n, $[\phi] = \bigcup_{p \in S^n(F_\gamma, \phi)} [\Phi_p]$.

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Omitting the measure 0 types from the sum,

$$\sum_{p \in S^n(F_\gamma,\mu)} \mu(p) = \mu(\phi) = 1.$$

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By the proposition, μ concentrates on a countable structure, which must be a model of ϕ. □

The end

Thank you!