Pseudofinite countably categorical theories

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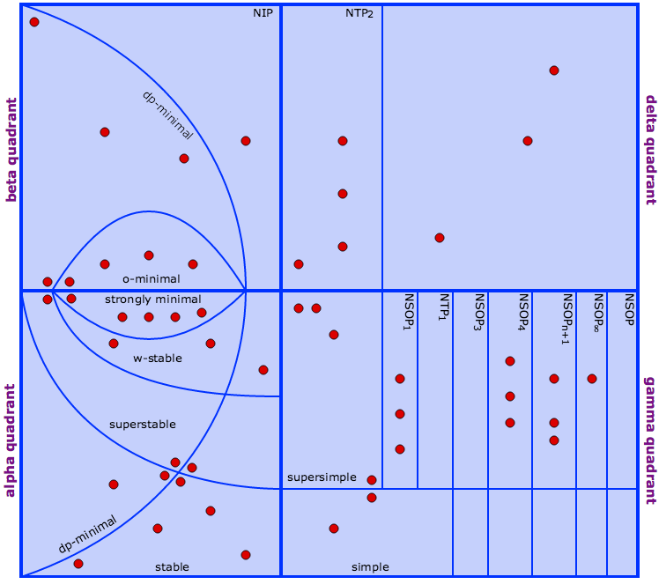
Question

Where do $\aleph_0$-categorical pseudofinite theories lie relative to model-theoretic dividing lines?

Map of the universe (Gabe Conant)



Pseudofiniteness and SOP

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By compactness, we can find a densely ordered chain $\{a_i \mid i \in [0, 1] \cap \mathbb{Q}\}$ with $a_i < a_j \leftrightarrow i < j$. By $\aleph_0$-categoricity, there is some formula $\varphi(x, y)$ such that $P \models \varphi(b, c)$ if and only if there is a densely ordered chain between b and c.

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Now $P \models (\exists x\, \exists y\, \varphi(x, y)) \land (\forall x\, \forall y\, \varphi(x, y) \to \exists z\, (x < z \land \varphi(z, y)))$.

This sentence cannot hold in any finite structure - it implies the existence of an infinite ascending chain. □

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Algebraic paradigm: Rigid structure. Easiest to show pseudofiniteness by exhibiting explicit finite models.

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Example: Infinite-dimensional vector space over $\mathbb{F}_p$.
Quasi-finitely axiomatizable: Theory of vector spaces + $\{\exists^{\geq n} x \top(x) \mid n \in \omega\}$.

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This example is one of a large family, the smoothly approximable structures, classified by Cherlin and Hrushovski (*Finite Structures with Few Types*). Built from pure sets and geometries coming from vector spaces, possibly with bilinear forms.

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Example: The random graph.
Axiomatizable by extension axioms: Theory of graphs +
$\{\forall \overline{x}\, \forall \overline{y}\, \exists z\, (\bigwedge_{i,j} x_i \neq y_j) \rightarrow (\bigwedge_{i=1}^{n} zEx_i) \wedge (\bigwedge_{j=1}^{n} \neg zEy_j) \mid n \in \omega\}$.

Let $\mathcal{G}(n)$ be the graphs with domain $[n] = \{1, \ldots, n\}$, and take μ_n to be the uniform measure on $\mathcal{G}(n)$. Then for any sentence φ in the theory of the random graph, $\lim_{n\to\infty} \mu_n(\{G \in \mathcal{G}(n) \mid G \models \varphi\}) = 1$.
We say that the theory of the random graph is the *almost-sure theory* of the sequence $(\mathcal{G}(n), \mu_n)_{n\in\omega}$.

So every sentence of this theory has a rich class of finite models: Almost all large finite graphs.

A brief review of Fraïssé theory

Let L be a relational language. A *Fraïssé class* is a class K of finite L-structures, closed under isomorphism, such that

1. K is countable up to isomorphism.
2. K has the hereditary property: If $B \in K$ and A embeds in B, then $A \in K$.
3. K has the joint embedding property and the amalgamation property:

$$\begin{tikzcd} & D & \\ A \arrow[ur, dashed] & & B \arrow[ul, dashed] \end{tikzcd} \qquad \begin{tikzcd} & D & \\ A \arrow[ur, dashed] & & B \arrow[ul, dashed] \\ & C \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Since we are interested in countably categorical theories, we assume that our Fraïssé classes contain only finitely many structures of size n up to isomorphism for each n.

A brief review of Fraïssé theory

Let K be a Fraïssé class. Then there is a unique countable structure up to isomorphism M_K, the *Fraïssé limit* of K, such that

1. K is the class of finite substructures of M_K.
2. M_K is homogeneous: Any isomorphism between finite substructures extends to an automorphism of M_K.

Let $T_K = \mathrm{Th}(M_K)$. Then T_K is $\aleph_0$-categorical, has quantifier elimination, and can be explicitly axiomatized by:

1. Universal axioms asserting that every finite substructure is in K.
2. A $\forall\exists$ *extension axiom* for every embedding $A \to B$ in K:

$$\forall \overline{x}\, \exists \overline{y}\, \theta_A(\overline{x}) \to \theta_B(\overline{x}, \overline{y})$$

where θ_X describes the quantifier-free type of X.

T_K is called the *generic theory* of K.

The canonical language

Let T be any $\aleph_0$-categorical theory and M its unique countable model.

- Let L^* be the language containing one n-ary predicate for each n-type relative to T. L^* is called the *canonical language* for T.
- Give M^* is natural L^*-structure.
- The class K_{T^*} of all finite substructures of M^* is a Fraïssé class, with Fraïssé limit M^* and generic theory T^*.

T and T^* are *interdefinable*.

Fraïssé Expansions

Definition

Let K and K' be Fraïssé classes in languages L and L', respectively, such that $L \subseteq L'$. We say that K is an expansion of K if

1. $K = \{A \upharpoonright L \mid A \in K'\}$
2. For all extensions (A, B) in K, and every expansion of A to a structure A' in K', there is an expansion of B to a structure B' in K' such that (A', B') is an extension.

$$\begin{tikzcd} A \arrow[r] & B \\ A' \arrow[u, no head, "\upharpoonright L"'] \arrow[r, dashed] & B' \arrow[u, dashed, no head, "\upharpoonright L"'] \end{tikzcd}$$

K' is an expansion of K if and only if the Fraïssé limit $M_{K'}$ of K' is an expansion of the Fraïssé limit M_K of K.

Disjoint n-amalgamation

Theorem (K.)

If a countably categorical theory T has disjoint n-amalgamation for all n, then T is pseudofinite. Consequently, any reduct of T is also pseudofinite.

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A type $p(\overline{x})$ over A is *non-redundant* if it does not contain the formulas $x_i = x_j$ for $i \neq j$ or $x_i = a$ for $a \in A$.

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A theory has *disjoint n-amalgamation* if whenever we have tuples of variables $\overline{x}_1, \ldots, \overline{x}_n$ and a system of types $\{p_S \mid S \subsetneq [n]\}$ over A such that p_S is a non-redundant type in the variables $\{\overline{x}_i \mid i \in S\}$, and $p_S \subseteq p_T$ when $S \subseteq T$, then there is some non-redundant type $p_{[n]}$ in the variables $\{\overline{x}_i \mid i \in [n]\}$ extending the p_S.

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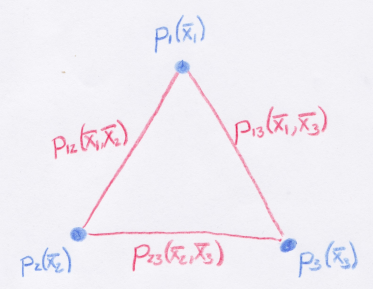
Fact

A countably categorical theory has disjoint 2-amalgamation if and only if it has trivial algebraic closure: acl$(A) = A$ for all sets A.

3-amalgamation

Given $p_i(\overline{x}_i)$ and $p_{ij}(\overline{x}_i, \overline{x}_j)$, $i, j \in \{1, 2, 3\}$, can we find $p_{123}(\overline{x}_1, \overline{x}_2, \overline{x}_3)$?



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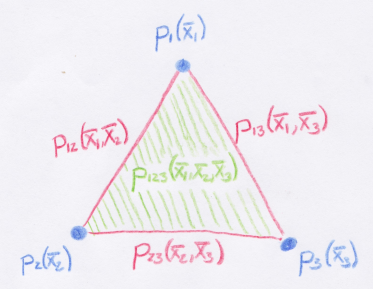

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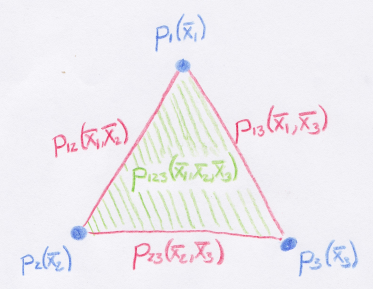

Examples of failure: Let T_K be the generic theory of a Fraïssé class K.

- $K =$ partial orders. $x_1 < x_2$, $x_2 < x_3$, $x_3 < x_1$.
- $K =$ equivalence relations. $x_1 E x_2$, $x_2 E x_3$, $\neg x_1 E x_3$.
- $K =$ triangle-free graphs. $x_1 E x_2$, $x_2 E x_3$, $x_1 E x_3$.

Probabilistic construction

Theorem (K.)

If a countably categorical theory T has disjoint n-amalgamation for all n, then T is pseudofinite. Consequently, any reduct of T is also pseudofinite.

Using the canonical language, we may assume that T is the generic theory of a Fraïssé class K. We describe a probability measure μ_n on $K(n)$, the structures in K with domain $[n] = \{1, \ldots, n\}$ by giving a probabilistic construction of such a structure. Then we check that T is the almost-sure theory of $(K(n), \mu_n)_{n \in \omega}$.

The measure μ_n is typically *not* the uniform measure on $K(n)$, but it is still very *natural*, and the probabilistic argument produces a large number of finite models in K for every sentence in T.

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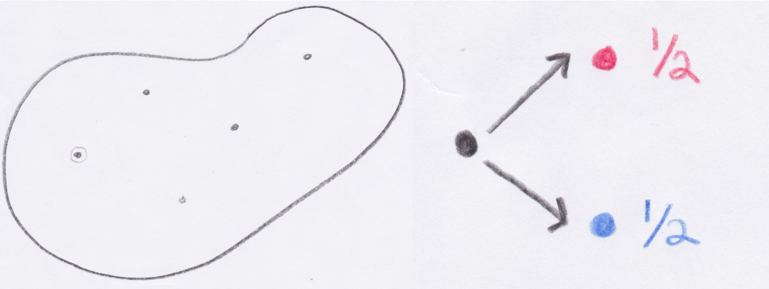
We build from below, assigning quantifier-free 1-types to each element, uniformly at random.



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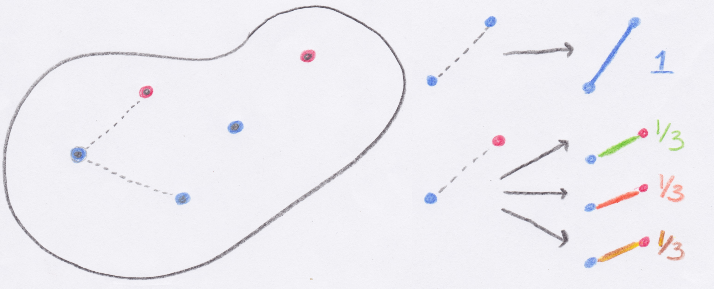
Then assign quantifier-free 2-types to each pair, uniformly among those extending the given 1-types.



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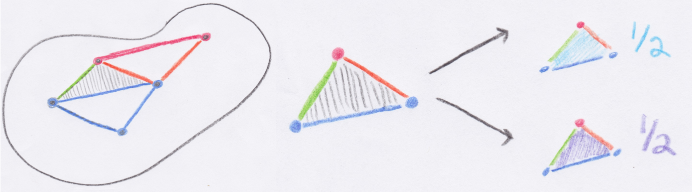
Disjoint n-amalgamation for all n ensures that we never get stuck.



... and that all choices were made as independently as possible. This allows us to verify that the extension axioms hold almost surely as n gets large.

An outrageous conjecture

In their paper *From stability to simplicity* (1998), Kim and Pillay offered the following "outrageous conjecture":

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Every pseudofinite $\aleph_0$-categorical theory is simple.

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This conjecture is false (as we shall see). However, we have:

Fact (Cherlin-Hrushovski)

All smoothly approximable theories are simple.

Fact (K.)

All reducts of theories with disjoint 2- and 3-amalgamation are simple.

Amalgamation and simplicity

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This follows easily from the characterization of forking in simple theories by Kim and Pillay.

Assuming disjoint 2- and 3-amalgamation, the disjointness relation

$$A \underset{C}{\downarrow} B \text{ iff } A \cap B \subseteq C$$

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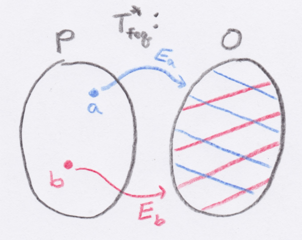
In fact, a countably categorical theory with disjoint n-amalgamation for all n will be ω-simple (n-simple for all n) in the sense of Kim, Kolesnikov, and Tsuboi.

Counterexample: T^*_{feq}

Our language has two sorts, O (objects), and P (parameters), and one ternary relation $E_x(y,z)$. K_{feq} is the class of finite structures with the property that for all $a \in P$, $E_a(y,z)$ is an equivalence relation on O.



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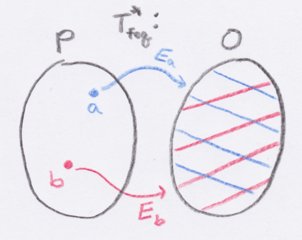

K_{feq} is a Fraïssé class. Its generic theory, T^*_{feq}, is not simple (Shelah) but is NSOP$_1$ (Chernikov and Ramsey).

Theorem (K.)

T^*_{feq} is pseudofinite.

The uniform measures

Let $K_{\mathsf{feq}}(n,m)$ be the structures in K_{feq} with O-sort $[n]$ and P-sort $[m]$, and let $\mu_{n,m}$ be the uniform measure on $K_{\mathsf{feq}}(n,m)$. T^*_{feq} is *not* the almost-sure theory of $(K_{\mathsf{feq}}(n,m), \mu_{n,m})$ as n and m grow.

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Fact (Flajolet and Sedgewick, Proposition VIII.8)

The expected number of equivalence classes in an equivalence relation on a set of size n, chosen uniformly, grows asymptotically as $\frac{n}{\log(n)}(1 + o(1))$.

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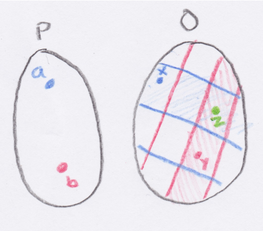

T^*_{feq} says:

$$\forall(a:P)\,\forall(b:P)\,\forall(x:O)\,\forall(y:O)\,\exists(z:O)\,((a \neq b) \to E_a(x,z) \wedge E_b(y,z))$$

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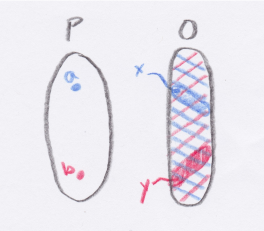

But asymptotically:

$$\forall(a : P)\,\forall(b : P)\,\forall(x : O)\,\forall(y : O)\,\exists(z : O)\,((a \neq b) \to E_a(x, z) \wedge E_b(y, z))$$

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Question

Does K_{feq} have a zero-one law for the uniform measures?

Filtered Fraïssé classes

Definition

A Fraïssé class K is *filtered* by a chain $K_0 \subseteq K_1 \subseteq K_2 \subseteq \dots$ if each K_n is a Fraïssé class, and $\bigcup_{n\in\omega} K_n = K$.

The generic theory T_K of K is the limit of generic theories T_{K_n}.
$\varphi \in T_K$ if and only if $\varphi \in T_{K_n}$ for all sufficiently large n.

Hence to show that T_K is pseudofinite, it suffices to show that each T_{K_n} is pseudofinite.

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Let K_n be the subclass of K_{feq} consisting of those structures in which the equivalence relation E_a has at most n classes, for each parameter $a \in P$.

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Let L_n be the expanded language obtained by adding n binary relations $R^1(x,y), \ldots, R^n(x,y)$. Then K_n admits an expansion to a Fraïssé class K_n', obtained by naming each E_a-equivalence class by one of the formulas $R^i(a,y)$ for each parameter $a \in P$.

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Each K_n has disjoint k-amalgamation for all k, and hence its generic theory T_{K_n} is pseudofinite. Thus T^*_{feq} is pseudofinite.

The Henson graph

We were able to push the "expansion with n-amalgamation" argument to some unsimple theories via the filtered Fraïssé class method. This method cannot apply to the Henson graph (the Fraïssé limit of the class of all triangle-free graphs).

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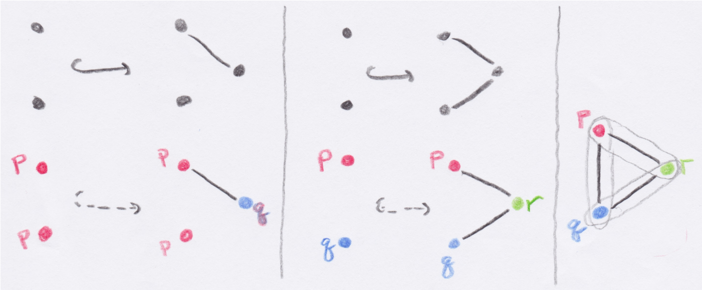
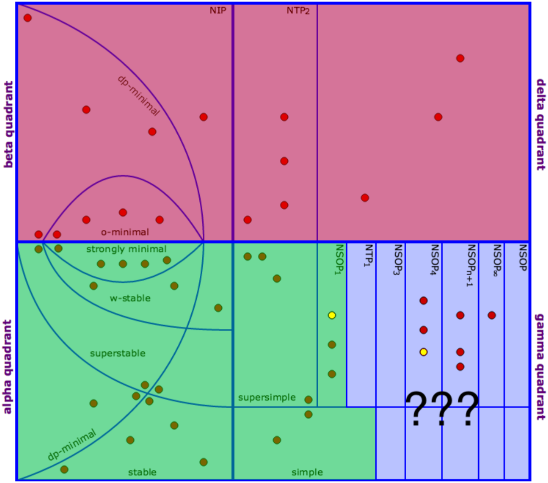


Map of the universe revisited



As far as I know, the question of pseudofinitenss is open for *every* $\aleph_0$-categorical theory in the blue region.

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References

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