All and Only

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pre-final version*

1 Introduction

This paper is a note on a very simple and elementary fact having to do with reasoning about relations. The kinds of reasoning that we have in mind come from two different areas: extended syllogistic logics and description logics. In extended syllogistic logic, one wants to reason about terms formed from binary relations and terms. We illustrate with some examples from the recent paper van Rooij [11]. He symbolizes "Some man loves every woman" (with the wide-scope reading: some man is such that he loves every woman) as $Mi(WaL)$. In this example, M and W are the terms for "man" and "woman", and L is the binary relation "loves." The i and a are from traditional logic: i for "some" and a for "all." So the term WaL is intended to denote the set of things who love all women. And then our sentence $Mi(WaL)$ would be true if some man belongs to the denotation of WaL. For us, the crucial point is that WaL has the semantics that we said it has. This kind of semantics is the source of later work, including presentations of logical calculi for the relational syllogistic in works such as [10].

We contrast this with the situation in description logics (e.g., [1]). In that field, we have atomic concepts such as W for woman, and binary roles such as L for loves. And then $\forall R.W$ would denote the set of objects with the property that everything which they love is a woman. To compare and contrast, and to anticipate notation which we shall use later in the paper, WaL is love all women, and $\forall R.W$ is love only women.

*This is not identical to the published version of our paper. That final version appears in H. Christiansen, M. D. Jiménez-López, R. Loukanova, and L. S. Moss (eds.), Proceedings, *Partiality and Underspecification in Information, Languages, and Knowledge*, Cambridge Scholars Publishing, 2017, 189–217.

Another source of the two different formalisms is modal logic. The most common semantics for modal logic is the relational (Kripke) semantics. One has a set W of "worlds", a relation that we are going to write as $[\![\mathsf{see}]\!]$ (think of it as the "seeing" relation in a model), and some way to interpret atomic propositions. The syntax of modal logic has a "box operator" $\Box$, and we shall see its semantics below. But there is a second, less-studied operator, $\boxminus$, suggestively called 'window'. An early reference for it is Gargov, Passy, and Tinchev [3]. Let us now compare the main clause in the semantics of both operators. To do this, we associate with the transitive verb see two operators, $\boxminus_{\mathsf{see}}$ and $\Box_{\mathsf{see}}$. They are defined by:

$$w \models \boxminus_{\mathsf{see}}\varphi \quad \text{iff} \quad v \models \varphi \text{ implies } w[\![\mathsf{see}]\!]v \qquad \text{"}w \text{ sees all } \varphi \text{ worlds"} \tag{1}$$

$$w \models \Box_{\mathsf{see}}\varphi \quad \text{iff} \quad w[\![\mathsf{see}]\!]v \text{ implies } v \models \varphi \qquad \text{"}w \text{ sees only } \varphi \text{ worlds"} \tag{2}$$

This paper originates in a talk presented to the European Summer School in Logic, Language, and Information (ESSLLI) in Bolzano, Italy in 2016. One of the many courses at ESSLLI was on description logics, taught by Uli Sattler and Thomas Schneider. Moss' talk mentioned logical calculi for the relational syllogistic which are the subject of Ian Pratt-Hartmann's and his paper [10], and thus the talk used syntax along the lines of r all x. And afterwards, Sattler asked why the talk treated relations the way it did. In effect, why did it not treat r only x? His "answer" was to say that the formulation r all x was more natural and more useful than r only x. Very soon after, he reconsidered this hasty reply. He came to the conclusion that both formulations are useful and important. Both deserve to be studied. This paper is thus a new formulation of the relational syllogistic. It raises some questions, answers a few of them, and leaves several unanswered. The main new results are logical rules and completeness theorems for logics employing the r only x construction.

Perhaps the most basic observation is that with negation on nouns and verbs (in description logic parlance, role complements as well as concept complements), the two formulations are inter-translatable. The point is that those who see only φ's are exactly those who fail-to-see all non-φs:

$$w \models \Box_{\mathsf{see}}\varphi \qquad \text{iff} \quad w \models \boxminus_{\overline{\mathsf{see}}}\, \neg\varphi \tag{3}$$

Now full negation on nouns (concepts) is relatively un-problematic in logics of the kinds we are interested in. Full negation on verbs (roles) is more subtle. It is associated with jumps in complexity in both syllogistic logics and description logics. In any case, the main point of this paper is to investigate settings where the two term-forming operators are not inter-translatable. So we shall study the setting without any form of negation, and also the setting where we have noun-level negation.

Another comment worth making is that both formulations would agree on the existential assertions. That is, the syllogistic WiL, the "natural logic" love some woman, and the description logic $\exists W.L$ all mean the same thing. And just as we are interested in combining the "all" terms and "some" terms, we are going to combine the "only" terms with the "some" terms.

Contents of this paper Our intent is to write a paper which is short and yet self-contained. Sections 2 and 3 re-present a few completeness theorems from earlier papers [6, 7, 8, 10]. Section 4 and onward is new material. Section 6.1 has the most difficult theorem in the paper, but the proof may be skipped without loss of continuity. Section 7 is mostly programmatic, stating open problems about logics that arise naturally in our study. Much of the thrust of that section is to re-consider results on the relational syllogistic from [10], but in the reformulation which this paper suggests.

2 Background: Syllogistic Logic of All and Some

We review here the most basic results in natural logic, the logical systems corresponding to sentences all p are q and some p are q.

Let $\mathbf{P}$ be a set of *nouns*. We first take a logical system $\mathcal{A}$ by take as syntactic objects the *sentences* all p are q for $p, q \in \mathbf{P}$. For the semantics, we use *models* $\mathcal{M}$. A model here consists of a set M (called *the universe* of $\mathcal{M}$) together with an interpretation function $[\![\ ]\!] : \mathbf{P} \to \mathcal{P}(M)$. (Here $\mathcal{P}(M)$ is the set of subsets of M.) Then we say that

$$\mathcal{M} \models \textsf{all } p \textsf{ are } q \quad \text{iff} \quad [\![p]\!] \subseteq [\![q]\!]. \tag{4}$$

We use φ for syntactic objects of whatever language is under discussion. For Γ a set of sentences in $\mathcal{A}$, we say that $\mathcal{M} \models \Gamma$ if $\mathcal{M} \models \varphi$ for each $\varphi \in \Gamma$. We then say that $\Gamma \models \varphi$ if for all $\mathcal{M}$, $\mathcal{M} \models \Gamma$ implies $\mathcal{M} \models \varphi$.

We have a logical system A defined using the following two rules:

$$\frac{}{\textsf{all } x \textsf{ are } x}\ \text{AXIOM} \qquad\qquad \frac{\textsf{all } x \textsf{ are } y \quad \textsf{all } y \textsf{ are } z}{\textsf{all } x \textsf{ are } z}\ \text{BARBARA} \tag{5}$$

A *proof tree over* Γ is a tree whose leaves are labeled with sentences in Γ, with the property that the non-leaves must match one of the rules in the system. We say that $\Gamma \vdash \varphi$ if there is a proof tree over Γ whose root is labeled φ.

The following is the simplest soundness/completeness theorem in logic:

Theorem 2.1 ([6]) *For all* $\Gamma \cup \{\varphi\}$ *in* $\mathcal{A}$, $\Gamma \vdash \varphi$ *in* A *iff* $\Gamma \models \varphi$.

Proof Fix Γ. The soundness direction is an easy induction on proof trees over Γ. For the completeness, assume that $\Gamma \models \varphi$. Let $\mathcal{M} = \mathcal{M}_\Gamma$ be the *canonical model* whose points are the nouns, and with

$$[\![p]\!] \quad = \quad \{q : \Gamma \vdash \textsf{all } q \textsf{ are } p\}. \tag{6}$$

One checks easily that $\mathcal{M} \models \Gamma$, using (BARBARA). Thus $\mathcal{M} \models \textsf{all } x \textsf{ are } y$, since this last sentence follows semantically from Γ. So $[\![x]\!] \subseteq [\![y]\!]$. But $x \in [\![x]\!]$, by the rule (AXIOM). Hence $x \in [\![y]\!]$. And this goes to show that $\Gamma \vdash \textsf{all } x \textsf{ are } y$, as desired. $\dashv$

We continue by adding to our syntax the sentences $\textsf{some } p \textsf{ are } q$ to our language. We shall call this language $\mathcal{AS}$ in this paper; $\mathcal{S}$ stands for $\textsf{some}$. (In [10], $\mathcal{S}$ refers to the language that also has $\textsf{no } p \textsf{ are } q$.) The semantics is given by

$$\mathcal{M} \models \textsf{some } p \textsf{ are } q \quad \text{iff} \quad [\![p]\!] \cap [\![q]\!] \neq \emptyset. \tag{7}$$

We construct a logical system AS using the rules of A, and also the rules below:

$$\begin{array}{c} \dfrac{\textsf{some } p \textsf{ are } q}{\textsf{some } q \textsf{ are } p} \ \textsc{some}_1 \\[2ex] \dfrac{\textsf{some } p \textsf{ are } q}{\textsf{some } p \textsf{ are } p} \ \textsc{some}_2 \\[2ex] \dfrac{\textsf{all } q \textsf{ are } n \quad \textsf{some } p \textsf{ are } q}{\textsf{some } p \textsf{ are } n} \ \textsc{darii} \end{array} \tag{8}$$

Theorem 2.2 ([6]) *For all* $\Gamma \cup \{\varphi\}$ *in* $\mathcal{AS}$, $\Gamma \vdash \varphi$ *in* AS *iff* $\Gamma \models \varphi$.

3 $\mathcal{A}(\mathcal{RC})$

The basic syllogistic logics which we recalled in Section 2 are not the focus of attention in this paper. What interests us more are several logical systems that go beyond syllogistic logics by employing *transitive verbs*. We refer to them as *binary atoms* in what follows.

Definition 3.1 *Here is the syntax of $\mathcal{A}(\mathcal{RC})$. We start with one collection* $\mathbf{P}$ *of* unary atoms *(for nouns), and with another collection,* $\mathbf{R}$ *of* binary atoms.

We define the terms *of $\mathcal{A}(\mathcal{RC})$ to be the smallest collection containing the unary atoms and with the property that if x is a term and r is a binary atom, then*

$$r \text{ all } x$$

is a term.

Note that terms allow recursion. So we get terms like

$$\text{see all (like all (hate all dogs)).} \tag{9}$$

(Of course, this is on the assumption that $\mathbf{P}$ *and* $\mathbf{R}$ *contain the words shown above as atoms.) The interpretation of the term above in a given model would be the set of individuals who see all who like all who hate all dogs.*

The sentences *of $\mathcal{A}(\mathcal{RC})$ are the expressions* all x y*, where x and y are terms. But please note that we have dropped the word* are *from sentences.*

The "$\mathcal{RC}$" in $\mathcal{A}(\mathcal{RC})$ stands for "relative clause."

We frequently use parentheses in the syntax to increase the readability.

Definition 3.2 *A* model $\mathcal{M}$ *for $\mathcal{A}(\mathcal{RC})$ is a set M, called (as before) the* universe*, together with interpretations of the atoms. For each unary atom p, we have an interpretation $[\![\mathrm{p}]\!] \subseteq M$. And for each binary atom r, we have an interpretation $[\![\mathrm{r}]\!] \subseteq M \times M$. (Here and below, $M \times M$ means the cartesian product of M with itself, the set of ordered pairs (x, y) such that both x and y belong to M.)*

We use recursion to interpret the terms. The model comes with interpretations of the unary atoms, the "base case" of terms. And the general case is

$$[\![r \text{ all } x]\!] \quad = \quad \{m \in M : \text{for all } n \in [\![x]\!],\ m[\![r]\!]n\}. \tag{10}$$

In other words, we are extending the interpretation function from the unary atoms to the set of all terms.

Finally, we have the definition of truth of sentences in a model: see (4).

Connections to modal logic Let us return to (1) and to see how to translate the current logic $\mathcal{A}(\mathcal{RC})$ into the modal logic of the $\boxminus$-operator and the *universal modality*. This last operator has the semantics

$$w \models U\varphi \quad \text{iff} \quad \text{for all } v,\ v \models \varphi. \tag{11}$$

Our unary atoms correspond to atomic propositions. Each binary atom r gives an operator $\boxminus_r$. Every term x of $\mathcal{A}(\mathcal{RC})$ corresponds to formula x^* in the modal logic with these window operators. For example, the term in (9) corresponds to

$$\boxminus_{\text{see}} \boxminus_{\text{like}} \boxminus_{\text{hate}} \text{ dogs.}$$

We continue the translation to sentences by

$$(\mathsf{all}\ x\ \mathsf{are}\ y)^* \quad = \quad U(x^* \to y^*).$$

This connection works on the semantic level, in the obvious way.

Logic We define *proof trees* using the rules below, and then we get the notion of $\Gamma \vdash \varphi$ just as in our previous work.

Definition 3.3 *The rules of* $\mathsf{A(RC)}$ *are* (AXIOM) *and* (BARBARA)*, repeated below, and also:*

$$\frac{}{\mathsf{all}\ x\ x}\ \text{AXIOM}$$

$$\frac{\mathsf{all}\ x\ y \quad \mathsf{all}\ y\ z}{\mathsf{all}\ x\ z}\ \text{BARBARA}$$

$$\frac{\mathsf{all}\ x\ (r\ \mathsf{all}\ y) \quad \mathsf{all}\ z\ y}{\mathsf{all}\ x\ (r\ \mathsf{all}\ z)}\ \text{ALL}$$

Note: x, y, and z may be any terms; they need not be unary atoms. On the other hand, r must be a binary atom.

We note that the following rule is derivable in the system:

$$\frac{\mathsf{all}\ y\ x}{\mathsf{all}\ (r\ \mathsf{all}\ x)\ (r\ \mathsf{all}\ y)}\ \text{ANTI}$$

Indeed, on top of (AXIOM) and (BARBARA), the rules (ALL) and (ANTI) are inter-derivable:

$$\frac{\dfrac{}{\mathsf{all}\ (r\ \mathsf{all}\ x)\ (r\ \mathsf{all}\ x)}\ \text{AXIOM} \qquad \mathsf{all}\ y\ x}{\mathsf{all}\ (r\ \mathsf{all}\ x)\ (r\ \mathsf{all}\ y)}\ \text{ALL}$$

$$\frac{\mathsf{all}\ x\ (r\ \mathsf{all}\ y) \qquad \dfrac{\mathsf{all}\ z\ y}{\mathsf{all}\ (r\ \mathsf{all}\ y)\ (r\ \mathsf{all}\ z)}\ \text{ANTI}}{\mathsf{all}\ x\ (r\ \mathsf{all}\ z)}\ \text{BARBARA}$$

The system is sound, and the main work is thus to prove the completeness.

Definition 3.4 *Given a set Γ of sentences in $\mathcal{A}(\mathcal{RC})$, we define the* canonical model $\mathcal{M} = \mathcal{M}_\Gamma$ *as follows: We take M to be the set of all terms. Let p be a unary atom and r a binary one. And then we define*

$$\begin{array}{rcl} [\![\mathrm{p}]\!] & = & \{x : \Gamma \vdash \mathsf{all}\ x\ p\} \\ [\![\mathrm{r}]\!] & = & \{(x, y) : \Gamma \vdash \mathsf{all}\ x\ (r\ \mathsf{all}\ y)\}. \end{array} \tag{12}$$

The clause for unary atoms is what we saw in (6). But the clause for the binary atoms should not be obvious.

Example 3.5 Suppose we have two unary atoms p and q, and one binary atom r. Let

$$\Gamma \quad = \quad \{\mathsf{all}\ p\ q\}.$$

To make the notation simpler, let us write p_0 for p, and p_{n+1} for r all p_n; we adopt similar notation for q. Then in the canonical model of Γ, we have $[\![p]\!] = \{p_0\}$, $[\![q]\!] = \{p_0, q_0\}$, and the interpretation of r is shown below:

$$\begin{tikzcd}
p_0 & p_1 \arrow[l] & p_2 \arrow[l] \arrow[dl] & p_3 \arrow[l] & p_4 \arrow[l] \arrow[dl] & \cdots \\
q_0 & q_1 \arrow[l] \arrow[ul] & q_2 \arrow[l] & q_3 \arrow[l] \arrow[ul] & q_4 \arrow[l] & \cdots
\end{tikzcd}$$

The way we got this structure was to start generating all of the proof trees over our set Γ. Whenever $\Gamma \vdash \mathsf{all}\ x\ (r\ \mathsf{all}\ y)$, we get an edge $x \to y$ For example,

$$\Gamma \vdash \mathsf{all}\ p_2\ p_2$$

i.e., $\Gamma \vdash \mathsf{all}\ p_2\ (r\ \mathsf{all}\ p_1)$. So we have $p_2 \to p_1$ above. And

$$\Gamma \vdash \mathsf{all}\ (r\ \mathsf{all}\ q)\ (r\ \mathsf{all}\ p).$$

This gives us the arrow $q_1 \to p_0$. Then based on the repetitive patterns in small examples, we drew the graph above.

It is fairly easy to see that the arrows above are *included* in the canonical model. To do this, we only would need to exhibit the derivations corresponding to the arrows present in the figure. But to show that no others arrows are present, one would have to do additional work. One way is to show by induction on proof trees over Γ that if $\Gamma \vdash \mathsf{all}\ x\ (r\ \mathsf{all}\ y)$, then there is an arrow $x \to y$ in the graph above. This is clear for the instances of (AXIOM), $\mathsf{all}\ p_{n+1}\ (r\ \mathsf{all}\ p_n)$ and $\mathsf{all}\ q_{n+1}\ (r\ \mathsf{all}\ q_n)$. Further, assume our result for a given proof tree $\mathcal{T}$, and consider what happens when we add an application of (ANTI) at the root. The root of $\mathcal{T}$ corresponds to one of the four types of arrows: $p_{n+1} \to p_n$, $p_{2n+2} \to q_{2n+1}$, $q_{n+1} \to q_n$, and $q_{2n+1} \to p_{2n}$. Applying (ANTI) to each of the sentences corresponding to these gives sentences which correspond to the arrows $p_{n+2} \to p_{n+1}$, $q_{2n+3} \to p_{2n+2}$, $q_{n+2} \to q_{n+1}$, and $p_{2n+2} \to q_{2n+1}$. We also have to do an induction step for attaching two trees using (BARBARA), but we omit those details.

We now return to our development, showing the completeness of the logic using the canonical model. The next lemma is the key result in this direction.

Lemma 3.6 *Let Γ be a set of sentences in $\mathcal{A}(\mathcal{RC})$, and let $\mathcal{M}$ be the canonical model of Γ. For all terms x,*

$$[\![x]\!] \quad = \quad \{y : \Gamma \vdash \textsf{all}\ y\ x\}. \tag{13}$$

Proof The proof is by induction on x. When x is a unary atom, our result is by the definition of our model. Assume that x is a term and that (13) holds for x; we prove that

$$[\![r\ \textsf{all}\ x]\!] \quad = \quad \{y : \Gamma \vdash \textsf{all}\ y\ (r\ \textsf{all}\ x)\}.$$

There are two directions. First, let $y \in [\![r\ \textsf{all}\ x]\!]$. By induction hypothesis and (AXIOM), $x \in [\![x]\!]$. So $(y, x) \in [\![r]\!]$. That is, $\Gamma \vdash \textsf{all}\ y\ (r\ \textsf{all}\ x)$. In the other direction, assume that $\Gamma \vdash \textsf{all}\ y\ (r\ \textsf{all}\ x)$. Suppose that $z \in [\![x]\!]$. Then by induction hypothesis, $\Gamma \vdash \textsf{all}\ z\ x$. Using (ALL),

$$\Gamma \vdash \textsf{all}\ y\ (r\ \textsf{all}\ z).$$

So $(y, z) \in [\![r]\!]$. This for all $z \in [\![x]\!]$ shows that $y \in [\![r\ \textsf{all}\ x]\!]$. $\dashv$

Lemma 3.7 $\mathcal{M} \models \Gamma$.

Proof Suppose that Γ contains $\textsf{all}\ u\ v$. To see that this sentence holds in $\mathcal{M}$, let $y \in [\![u]\!]$. By Lemma 3.6, $\Gamma \vdash \textsf{all}\ y\ u$. And then using (BARBARA), we have $\Gamma \vdash \textsf{all}\ y\ v$. Therefore, $y \in [\![v]\!]$, just as desired. $\dashv$

Lemma 3.8 *If $\mathcal{M} \models \varphi$, then $\Gamma \vdash \varphi$.*

Proof Let φ be $\textsf{all}\ a\ b$. Then $[\![a]\!] \subseteq [\![b]\!]$ in $\mathcal{M}$. But $a \in [\![a]\!]$, using (AXIOM) in our system and also Lemma 3.6. And so $a \in [\![b]\!]$. By Lemma 3.6 again, we have $\Gamma \vdash \textsf{all}\ a\ b$. $\dashv$

Theorem 3.9 *The logical system* $\textsf{A(RC)}$ *is sound and complete for $\mathcal{A}(\mathcal{RC})$.*

Proof We have observed that the system is sound. So suppose $\Gamma \models \varphi$. By Lemma 3.7, $M \models \Gamma$, so $M \models \varphi$. So by Lemma 3.8, $\Gamma \vdash \varphi$. $\dashv$

The material in this section comes from [8].

4 $\mathcal{O}(\mathcal{RC})$: Another Addition to $\mathcal{A}$

Recall from our Introduction that this paper is about logics which allow terms to be formed using the construction r only x.

Definition 4.1 *$\mathcal{O}(\mathcal{RC})$ is the logical system defined the same way as $\mathcal{A}(\mathcal{RC})$ was, except that we include terms of the form*

$$r \text{ only } x$$

rather than r all x. Just as in $\mathcal{A}(\mathcal{RC})$, the sentences *of $\mathcal{O}(\mathcal{RC})$ are the expressions* all *x y, where x and y are terms.*

Terms again may use recursion. So we get terms like

$$\text{see only (like only (hate only dogs)).} \tag{14}$$

(Of course, this is on the assumption that **P** and **R** which underlie the language contain the words shown above.) The interpretation of the term above in a given model would be the set of individuals who see only those individuals who like only the individuals who hate only dogs.

We frequently use parentheses in the syntax to increase the readability. We prove facts by induction on terms in the usual way.

Definition 4.2 *A* model *$\mathcal{M}$ for $\mathcal{O}(\mathcal{RC})$ is a set M, called (standardly) the* universe*, together with interpretations of the atoms. For each unary atom p, we have an interpretation $[\![\mathrm{p}]\!] \subseteq M$. And for each binary atom r, we have an interpretation $[\![\mathrm{r}]\!] \subseteq M \times M$. We again use recursion to interpret the terms. Instead of (10), we use*

$$[\![r \text{ only } x]\!] \quad = \quad \{m \in M : \textit{for all } n \textit{ if } m[\![\mathrm{r}]\!]n, \textit{ then } n \in [\![\mathrm{x}]\!]\}. \tag{15}$$

The reader will not be surprised that the definition of truth of sentences in a model uses (4), as before.

Remark We are aware that the English word "only" presents a number of challenges to the syntactician and the semanticist. In the syntax, "only" differs from determiners like "all" and "some" because it combines much more freely. For example "only" may appear before each word in the sentence *Carmen loved José yesterday*, and this contrasts with the determiners. Moreover, all of the meanings are different. And for the semantics, "only" gives rise to Gricean implicatures and also focus effects. None one of this will be relevant in this paper. The main thing for us is that our semantics of terms is reasonable and gets at an interesting aspect of the meaning of "only" in the kinds of relative clauses we consider.

We now return to our main topic: the logic $\mathcal{O}(\mathcal{RC})$ and its semantics.

Connections to modal logic As before, we have a connection to modal logic, this time to the standard logic and its semantics. For example, (14) corresponds to

$$\Box_{\mathrm{see}}\Box_{\mathrm{like}}\Box_{\mathrm{hate}}\ \text{dogs}.$$

Logic Next, we have the parallel notion of logic, this time called $\mathsf{O(RC)}$. The rules are shown just below.

$$\frac{}{\mathsf{all}\ x\ x}\ \textsc{axiom}$$

$$\frac{\mathsf{all}\ x\ y \quad \mathsf{all}\ y\ z}{\mathsf{all}\ x\ z}\ \textsc{barbara}$$

$$\frac{\mathsf{all}\ x\ (\mathsf{r}\ \mathsf{only}\ y) \quad \mathsf{all}\ y\ z}{\mathsf{all}\ x\ (r\ \mathsf{only}\ z)}\ \textsc{only}$$

Note: x, y, and z may be any terms; they need not be unary atoms. And again, r may be any of our binary atoms.

As with $\mathcal{A}(\mathcal{RC})$, we have a derivable rule (mono) which is inter-derivable with (only) on top of (axiom) and (barbara).

$$\frac{\mathsf{all}\ x\ y}{\mathsf{all}\ (r\ \mathsf{only}\ x)\ (r\ \mathsf{only}\ y)}\ \textsc{mono}$$

The canonical model construction is different.

Definition 4.3 *Given a set Γ, we define a model $\mathcal{M} = \mathcal{M}_\Gamma$ in several steps. First, we define a relation $\leq$ on terms by*

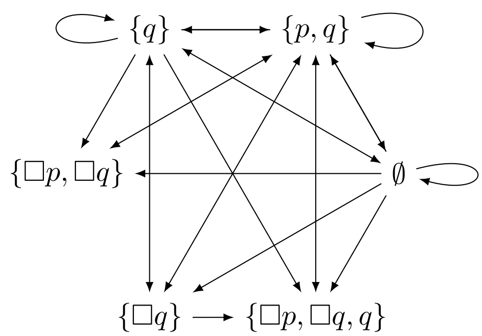
$$s \leq t \quad \textit{iff} \quad \Gamma \vdash \mathsf{all}\ s\ t. \tag{16}$$

*The relation $\leq$ is a preorder, by (*axiom*) and (*barbara*). We take M to be the set of all sets $\mathcal{S}$ of terms which happen to be up-closed in this preorder. And then we define*

$$\begin{array}{rcl} [\![\mathrm{p}]\!] & = & \{\mathcal{S} \in M : p \in \mathcal{S}\} \\ [\![\mathrm{r}]\!] & = & \{(\mathcal{S}, \mathcal{T}) \in M \times M : \textit{if } (r\ \mathsf{only}\ x) \in \mathcal{S},\ \textit{then } x \in \mathcal{T}\}. \end{array} \tag{17}$$

As always, p is a unary atom and r a binary one.

Example 4.4 We revisit $\Gamma = \{\mathsf{all}\ p\ q\}$ from Example 3.5, but this time as a set with a sentence in $\mathcal{O}(\mathcal{RC})$. To simplify matters, we assume that the only unary atoms are p and q, and that the only binary atom is r. The points in the canonical model are the sets of sentences with the property that for all n, if $r\ \mathsf{only}^n\ p$ belongs to $\mathcal{S}$, then so does $r\ \mathsf{only}^n\ q$. This model is uncountable. Here are six of its points, together with the arrows (interpreting r) between them.



We have written r only p as $\Box p$, and similarly for q. The interpretations of p and q are by (17). We remind the reader that the canonical model is actually uncountable, and so the picture is just a tiny piece.

Lemma 4.5 (Truth Lemma) *Let Γ be a set of sentences in $\mathcal{O}(\mathcal{RC})$, and let $\mathcal{M}$ be the canonical model of Γ. For all terms x,*

$$[\![x]\!] \quad = \quad \{\mathcal{S} \in M : x \in \mathcal{S}\}. \tag{18}$$

Proof By induction on x. For x atomic, our lemma follows immediately from Definition 4.3. We assume (18) for x and prove it for the term r only x.

First, let $\mathcal{S} \in M$ be such that $r\ \mathsf{only}\ x \in \mathcal{S}$. Let $\mathcal{T}$ be any element such that $\mathcal{S}[\![r]\!]\mathcal{T}$. We show that $\mathcal{T} \in [\![x]\!]$. By definition of $[\![r]\!]$, $x \in \mathcal{T}$. By induction hypothesis, we indeed have $\mathcal{T} \in [\![x]\!]$.

In the other direction, let $\mathcal{S} \in M$ be such that $r\ \mathsf{only}\ x \notin \mathcal{S}$. We exhibit some $\mathcal{T}$ such that $\mathcal{S}$ is related to it by $[\![r]\!]$, but with $\mathcal{T} \notin [\![x]\!]$. Let

$$\mathcal{T} \quad = \quad \{y : (r\ \mathsf{only}\ y) \in \mathcal{S}\}. \tag{19}$$

To see that $\mathcal{T}$ is up-closed, suppose $y \in \mathcal{T}$ (so r only $y \in \mathcal{S}$) and $y \leq z$. Then $\Gamma \vdash \mathsf{all}\ y\ z$, so by the rule (MONO), $\Gamma \vdash \mathsf{all}\ (r\ \mathsf{only}\ y)\ (r\ \mathsf{only}\ z)$. Now since $\mathcal{S}$ is up-closed, r only $z \in \mathcal{S}$, and hence $z \in \mathcal{T}$. By construction, $\mathcal{S}[\![r]\!]\mathcal{T}$. Suppose towards a contradiction that $\mathcal{T} \in [\![x]\!]$. By induction hypothesis, $x \in \mathcal{T}$. Thus r only $x \in \mathcal{S}$; this is a contradiction. ⊣

Lemma 4.6 $\mathcal{M} \models \Gamma$.

Proof Suppose that $\mathsf{all}\ x\ y$ belongs to Γ. We claim that $[\![x]\!] \subseteq [\![y]\!]$. Let $\mathcal{S} \in [\![x]\!]$. By Lemma 4.5, $x \in \mathcal{S}$. Since $\mathcal{S}$ is up-closed, $y \in \mathcal{S}$ also. And so by Lemma 4.5 again, $\mathcal{S} \in [\![y]\!]$. This proves our claim. ⊣

Theorem 4.7 (Completeness) *If* $\Gamma \models \varphi$, *then* $\Gamma \vdash \varphi$.

Proof Consider the canonical model $\mathcal{M}$ of Γ. We know that $\mathcal{M}$ satisfies Γ. Since $\Gamma \models \varphi$, we also have $\mathcal{M} \models \varphi$. Suppose φ is $\mathsf{all}\ x\ y$. Then $[\![x]\!] \subseteq [\![y]\!]$ in $\mathcal{M}$. The upward closure of $\{x\}$, $\uparrow x$, is an element of M, and it belongs to $[\![x]\!]$ by Lemma 4.5. Hence $\uparrow x \in [\![y]\!]$, and we see that $y \in \uparrow x$. And this means that $\Gamma \vdash \mathsf{all}\ x\ y$, as desired. ⊣

4.1 First Refinement: The Finite Model Property and Efficient Proof Search

Let $\Gamma \cup \{\varphi\}$ be a finite set of $\mathcal{O}(\mathcal{RC})$ sentences. If $\Gamma \not\vdash \varphi$, then the canonical model $\mathcal{M}_\Gamma$ of Γ does not satisfy φ. This model is infinite, indeed it is uncountable. In pursuit of an algorithmic treatment, let us show that there is a *finite* model of Γ where φ fails. Our work also shows that the relation $\Gamma \vdash \varphi$ is decidable in polynomial time.

Let $\mathbb{T}$ be the set of all subterms of terms appearing in sentences in $\Gamma \cup \{\varphi\}$. So $\mathbb{T}$ is a finite set of terms closed under subterms. We write $\Gamma \vdash_{\mathbb{T}} \psi$ if there is a derivation of ψ from Γ, such that all terms appearing in sentences in the proof tree belong to $\mathbb{T}$. Note that if $\Gamma \vdash_{\mathbb{T}} \psi$, then the subterms of ψ belong to $\mathbb{T}$.

We also write $\leq_{\mathbb{T}}$ for the relation defined by

$$x \leq_{\mathbb{T}} y \quad \text{iff} \quad \Gamma \vdash_{\mathbb{T}} \text{all } x\ y$$

Let N be the set of all subsets $\mathcal{S}$ of $\mathbb{T}$ with the following property: if $x \in \mathcal{S}$ and $\Gamma \vdash_{\mathbb{T}} \mathsf{all}\ x\ y$, then $y \in \mathcal{S}$. In words, $\mathcal{S}$ is closed upwards in $\leq_{\mathbb{T}}$. We define a model $\mathcal{N} = \mathcal{N}_{\Gamma,\varphi}$ using this set N and (17) for the structure.

Lemma 4.8 (Truth Lemma for $\mathcal{N}_{\Gamma,\varphi}$) *For all $x \in \mathbb{T}$,*

$$[\![x]\!] \quad = \quad \{\mathcal{S} \in N : x \in \mathcal{S}\}. \tag{20}$$

Proof By induction on $x \in \mathbb{T}$. The proof is nearly the same as that of Lemma 4.5. The only difference is in the inductive step for a term r only x in $\mathbb{T}$. Fix $\mathcal{S} \in N$, and suppose that this term does not belong to $\mathcal{S}$. We find some $\mathcal{T} \in N$ such that $\mathcal{S}[\![r]\!]\mathcal{T}$, but $x \notin \mathcal{T}$. We use

$$\mathcal{T} \quad = \quad \uparrow\{y \in \mathbb{T} : \text{for some } z \in \mathcal{S}, \Gamma \vdash_{\mathbb{T}} \mathsf{all}\ z\ (r\ \mathsf{only}\ y)\}.$$

The arrow denotes the upward-closed in $\leq_{\mathbb{T}}$, so this set $\mathcal{T}$ is in N by definition. We claim that $x \notin \mathcal{T}$. For if $x \in \mathcal{T}$, then there is some $z \in \mathcal{S}$ and some y such that $y \leq_{\mathbb{T}} x$, and also $\Gamma \vdash_{\mathbb{T}} \mathsf{all}\ z\ (r\ \mathsf{only}\ y)$. Putting the derivations together with one application of (ONLY) at the root, we have $\Gamma \vdash_{\mathbb{T}} \mathsf{all}\ z\ (r\ \mathsf{only}\ x)$. (Note that we are assuming that r only x belongs to $\mathbb{T}$.) And since $\mathcal{S}$ is upward-closed in $\leq_{\mathbb{T}}$, it contains r only x; this is a contradiction. ⊣

Here is a summary of our work in this section.

Theorem 4.9 *Let $\Gamma \cup \{\varphi\}$ be a finite set of $\mathcal{O}(\mathcal{RC})$ sentences. Let $\mathbb{T}$ be the finite set of terms in $\Gamma \cup \{\varphi\}$, together with their subterms. Then $\Gamma \vdash \varphi$ iff $\Gamma \vdash_{\mathbb{T}} \varphi$. Moreover, the problem of whether or not $\Gamma \vdash \varphi$ may be answered in time polynomial in the length of $\Gamma \cup \{\varphi\}$.*

Proof If $\Gamma \vdash_{\mathbb{T}} \varphi$, then obviously $\Gamma \vdash \varphi$. Suppose that $\Gamma \not\vdash_{\mathbb{T}} \varphi$; we show that the canonical model $\mathcal{N} = \mathcal{N}_{\Gamma,\varphi}$ is a (finite) model of Γ where φ fails. So by soundness of the logic, $\Gamma \not\vdash \varphi$.

To check that $\mathcal{N} \models \Gamma$, suppose that the sentence all a b belongs to Γ. Then a and b belong to $\mathbb{T}$, and $a \leq_{\mathbb{T}} b$. Hence every $\mathcal{S} \in N$ which contains a also contains b. And by the Truth Lemma 4.8, $\mathcal{N} \models \mathsf{all}\ a\ b$.

Let us write φ as all u v. Note that $u, v \in \mathbb{T}$. Let $\mathcal{S} =\uparrow u$. Then $\mathcal{S} \in N$. Recall that $\Gamma \not\vdash_{\mathbb{T}} \mathsf{all}\ u\ v$. Thus $v \notin \mathcal{S}$. So by the Truth Lemma 4.8, $\mathcal{S} \in [\![u]\!]$, but $\mathcal{S} \notin [\![v]\!]$. $\mathcal{S}$ shows that φ is false in $\mathcal{N}$.

This concludes the proof that $\Gamma \vdash \varphi$ iff $\Gamma \vdash_{\mathbb{T}} \varphi$. For the complexity estimate at the end, starting with $\Gamma \cup \{\varphi\}$, we find $\mathbb{T}$ and then we find $X = \{\psi : \Gamma \vdash_{\mathbb{T}} \psi\}$. The point is that X is the least fixed point of a monotone induction, and so it may be found in polynomial time, too. (This situation contrasts with logical systems which employ rules like cut or even *reductio ad absurdum*; for such systems, the derivability relation is typically not in polynomial time.) ⊣

4.2 Towards a Second Refinement: An Efficient Counter-Model Algorithm

Let $\Gamma \cup \{\varphi\}$ be a finite set with $\Gamma \not\vdash \varphi$. Theorem 4.9 shows that there is a finite model $\mathcal{N}_{\Gamma,\varphi}$ of Γ where φ fails. The proof of the theorem only gives a model whose size is exponential in the number of terms in $\Gamma \cup \{\varphi\}$. In order to get an feasible algorithmic treatment, we would also like to build countermodels more efficiently. So more work is needed on this point. Based on some examples, we conjecture that if $\Gamma \not\vdash \varphi$ in this logic, then there is a model of Γ where φ fails, and the size of the model is polynomial in Γ, φ. However, we have not been able to settle this matter the way we were with the logic of all and verbs.

We did not include in this paper a parallel treatment of the work in this section for the logic $\mathcal{A}(\mathcal{RC})$. However, such work has been done; see [8]. The analog of Theorem 4.9 is fairly easy to obtain. And if $\Gamma \not\vdash \varphi$, then the size of the canonical model of Γ which falsifies φ is polynomial in the size of $\Gamma \cup \{\varphi\}$.

5 Adding Term Formers r some x to $\mathcal{A}(\mathcal{RC})$

At this point, we return to $\mathcal{A}(\mathcal{RC})$. Let us expand our language of terms to allow terms r some x. Again, we allow recursion, so we also get terms like $\mathsf{see\ some\ (love\ all\ tigers)}$. The semantics is again by recursion, and we use (10) and also

$$[\![r \text{ } \mathsf{some} \text{ } x]\!] \quad = \quad \{m \in M : \text{for some } n \in [\![x]\!], \, m[\![r]\!]n\}. \tag{21}$$

Turning to the logic, we would add to $\mathsf{A(RC)}$ the following rules:

$$\begin{gathered}
\frac{\mathsf{all}\ x\ y}{\mathsf{all}\ (r\ \mathsf{some}\ x)\ (r\ \mathsf{some}\ y)} \\[1ex]
\frac{\mathsf{some}\ x\ y}{\mathsf{all}\ (r\ \mathsf{all}\ x)(r\ \mathsf{some}\ y)} \\[1ex]
\frac{\mathsf{some}\ y\ (r\ \mathsf{some}\ x)}{\mathsf{some}\ x\ x}
\end{gathered} \tag{22}$$

However, this is not enough. We go beyond purely syllogistic logic by adding a rule of *proof by cases*. In a natural deduction formulation, this would be:

$$\frac{\dfrac{[\mathsf{some}\ x\ x]}{\varphi} \qquad \dfrac{[\mathsf{all}\ x\ y \quad \mathsf{all}\ y\ (r\ \mathsf{all}\ x)]}{\varphi}}{\varphi}\ \text{CASES}$$

That is, to prove φ (from some set of assumptions), it is enough to both prove it from the assumption that there are x's, or from the opposite assumption. And this opposite assumption gets expressed by the second and third alternatives above. Now we do not have $\mathsf{no}\ x\ x$ in the language, but if we prove it from any sentence that is in the language and yet follows semantically from "there are no x" – such as $\mathsf{all}\ y\ (r\ \mathsf{all}\ x)$, then we may conclude φ outright. The use of the brackets in our statement means that we have temporary assumptions which are discharged (set in brackets). The soundness of this proof system would then say that if Γ includes the the un-bracketed leaves in a proof tree $\mathcal{T}$, then every model of Γ is a model of the root of $\mathcal{T}$.

The completeness proof may be found in [7]. We are not going to reprove this result here, since the details are rather different from what we need in our work below. We should mention that the fragment under current discussion is quite close to the language studied by McAllester and Givan [5]. Their paper also proves that the satisfiability problem for this logic is NP-complete.

6 Adding Sentences $\mathsf{some}\ x\ y$ to $\mathcal{O}(\mathcal{RC})$

In this section, we add to the syntax the sentences $\mathsf{some}\ x\ y$, with semantics as in (7). We can also add to the proof theory the syllogistic rules of some in (8). However, this is not enough: to prove the completeness theorem, we also need a rule scheme we call (R). As it happens, (R) has as special cases the rules (SOME$_1$), (SOME$_2$), and (DARII) from (8) (see Example 6.1 below). So the logical system of this section consists of the rules (AXIOM), (BARBARA), (ONLY), and the infinite scheme (R).

To state the rules of (R), and for use in the rest of this paper, we introduce some notation. Let $\vec{r} = r_1, \ldots, r_k$ be a sequence of binary atoms. This sequence may have repeated elements. For every term a, we define

$$\vec{r}\ \mathsf{only}\ a \quad = \quad r_1\ \mathsf{only}\ (r_2\ \mathsf{only} \cdots (r_n\ \mathsf{only}\ a) \cdots)$$

If $k = 0$, then the sequence $\vec{r}$ is the empty sequence. In this case, we take $\vec{r}\ \mathsf{only}\ a$ to be a itself. Further, we extend our semantic relation $[\![r]\!]$ in each model $\mathcal{M}$ by writing

$$[\![\vec{r}]\!] \quad = \quad [\![r_1]\!]; [\![r_2]\!]; \cdots ; [\![r_n]\!]$$

where the semicolon ; denotes relational composition.

The scheme (R) Fix terms x and y. Let $\varphi_0, \varphi_1, \ldots, \varphi_m$ be a sequence of sentences with the properties listed below.

1. φ_0 is of the form $\textsf{some}\ a\ b$.
2. For $1 \leq i \leq m$, φ_i is of the form $\textsf{all}\ u_i\ v_i$.
3. Each u_i has one of the following properties:
   (a) u_i is in $\{a, b, v_1, \ldots, v_{i-1}\}$.
   (b) There is a sequence $\vec{r}$ and a term z such that $\vec{r}\ \textsf{only}\ x$ and $\vec{r}\ \textsf{only}\ y$ are in $\{a, b, v_1, \ldots, v_{i-1}\}$, and u_i is $\vec{r}\ \textsf{only}\ z$.
4. x and y are in $\{a, b, v_1, \ldots, v_m\}$.

Then from $\varphi_0, \varphi_1, \ldots, \varphi_m$, infer $\textsf{some}\ x\ y$.

Remark One arrives at such a complicated scheme in the course of proving the completeness theorem. So to understand this rule, one would do well to study the proof of Theorem 6.6, especially the first claim in Section 6.1.

Example 6.1 We recover the syllogistic rules of (SOME) given in (8) from the scheme (R). Note that (SOME$_1$) and (SOME$_2$) are instances of the scheme (R) in the case $m = 0$. For (DARII), take ψ to be $\textsf{some}\ x\ y$, and take φ_1 to be $\textsf{all}\ y\ z$. Then (R) applies, and we conclude $\textsf{some}\ x\ z$.

We now embark on proving the soundness of (R).

Definition 6.2 *Let x and y be terms, and let Γ be a set of sentences. An* xy-sequence for Γ *is a sequence of terms*

$$t_1, \ldots, t_k$$

with $k \geq 2$, such that for all $i > 2$, one of the following holds:

1. *There is $j < i$ such that $\Gamma \vdash \textsf{all}\ t_i\ t_j$.*
2. *There are $j, k < i$, a sequence $\vec{r}$, and a term z, such that t_j is $\vec{r}\ \textsf{only}\ x$, t_k is $\vec{r}\ \textsf{only}\ y$, and t_i is $\vec{r}\ \textsf{only}\ z$.*

*If $\mathcal{M}$ is a model, an xy-*sequence in $\mathcal{M}$ *is an xy-sequence for* $\mathrm{Th}(\mathcal{M})$*, where* $\mathrm{Th}(\mathcal{M})$ *is the set of all sentences true in $\mathcal{M}$. Note that for all $\mathcal{M}$,* $\mathrm{Th}(\mathcal{M}) \vdash \textsf{all}\ t\ u$ *if and only if $[\![t]\!] \subseteq [\![u]\!]$ in $\mathcal{M}$.*

Lemma 6.3 *Let $t_1, \ldots, t_k$ be an xy-sequence in $\mathcal{M}$. Then either $[\![x]\!] \cap [\![y]\!] \neq \emptyset$, or else for all i, $[\![t_1]\!] \cap [\![t_2]\!] \subseteq [\![t_i]\!]$.*

Proof By induction on the length k of the sequence $t_1, \ldots, t_k$. For $k = 2$, the result is obvious. Assume our result for all xy-sequences of length k, and consider an xy-sequence $t_1, \ldots, t_k, t_{k+1}$. Then $t_1, \ldots, t_k$ is an xy-sequence of length k, and our induction hypothesis applies. If $[\![x]\!] \cap [\![y]\!] \neq \emptyset$, we are done. So we may assume that for all $i \leq k$, $[\![t_1]\!] \cap [\![t_2]\!] \subseteq [\![t_i]\!]$. Consider t_{k+1}. If for some $j < k+1$, $[\![t_j]\!] \subseteq [\![t_{k+1}]\!]$, our result is obvious. We thus assume that t_{k+1} is $\vec{r}$ only z where $i, j \leq k$ are such that t_i is $\vec{r}$ only x and t_j is $\vec{r}$ only y. Assume first that for some $m \in [\![t_1]\!] \cap [\![t_2]\!]$, m is related by $[\![\vec{r}]\!]$ to some m'. Then, since $m \in [\![\vec{r} \text{ only } x]\!] \cap [\![\vec{r} \text{ only } y]\!]$, this m' belongs to $[\![x]\!] \cap [\![y]\!]$, and we are done. If this fails, then every $m \in [\![t_1]\!] \cap [\![t_2]\!]$ is not related by $[\![\vec{r}]\!]$ to anything. So each such m is in $[\![\vec{r} \text{ only } z]\!]$ (for all z). ⊣

As an immediate consequence, we have the following fact:

Lemma 6.4 *Let $t_1, \ldots, t_k$ be an xy-sequence in $\mathcal{M}$. Suppose that $[\![t_1]\!] \cap [\![t_2]\!] \neq \emptyset$, and also that x and y themselves occur in the sequence $t_1, \ldots, t_k$. Then $[\![x]\!] \cap [\![y]\!] \neq \emptyset$.*

Proof By Lemma 6.3, either $[\![x]\!] \cap [\![y]\!] \neq \emptyset$, or else $[\![t_1]\!] \cap [\![t_2]\!] \subseteq [\![x]\!] \cap [\![y]\!]$. In the second case, the hypothesis that $[\![t_1]\!] \cap [\![t_2]\!] \neq \emptyset$ yields the desired conclusion. ⊣

Proposition 6.5 *Every instance of the scheme* (R) *is sound.*

Proof Fix an instance of (R), and use the notation from the statement of the rule. Fix a model $\mathcal{M}$ in which the premises are true. Recall that sentences true in $\mathcal{M}$ are provable from $Th(\mathcal{M})$. We claim that the sequence

$$a, b, u_1, v_1, \ldots, u_m, v_m$$

is an xy-sequence in $\mathcal{M}$. For each u_i, either u_i occurs earlier in the sequence, or u_i is among $\{a, b, v_1, \ldots, v_{i-1}\}$ (and note that all u_i u_i is true in $\mathcal{M}$), or u_i is of the form $\vec{r}$ only z such that $\vec{r}$ only x and $\vec{r}$ only y appear earlier in the sequence. And for each v_i, all u_i v_i is true in $\mathcal{M}$.

Further, $[\![a]\!] \cap [\![b]\!] \neq \emptyset$, since some a b is true in $\mathcal{M}$, and x and y appear in the sequence, among $\{a, b, v_1, \ldots, v_m\}$. So by Lemma 6.4, the sentence some x y holds in $\mathcal{M}$. ⊣

The main work in this section is to prove the following result.

Theorem 6.6 (Completeness) *If* $\Gamma \models \varphi$*, then* $\Gamma \vdash \varphi$*.*

We have two cases depending on the shape of φ, and we indicate the divisions by subsections in our text. Just below, we take care of the case that φ is an all sentence, giving a direct reduction to the work in Section 4. The work following that is for the some sentences; it is more involved.

The Proof when $\Gamma \models \mathsf{all}\ x\ y$ In this case, we build the canonical model $\mathcal{M}$ for Γ exactly as in Definition 4.3. The proof of completeness for all sentences (Theorem 4.7) goes through almost without change. We only need to adjust Lemma 4.6, to check that $\mathcal{M}$ satisfies the sentences in Γ of the form $\mathsf{some}\ a\ b$. Letting $\mathcal{S} = (\uparrow a) \cup (\uparrow b)$, we see that $\mathcal{S}$ is up-closed, so $\mathcal{S} \in M$, and $\mathcal{S} \in [\![a]\!] \cap [\![b]\!]$ by Lemma 4.5.

The Proof when $\Gamma \not\vdash \mathsf{some}\ x\ y$. In this case, we will prove the contrapositive of completeness. That is, assuming that $\Gamma \not\vdash \mathsf{some}\ x\ y$, we shall build a model $\mathcal{M}$ of Γ where $[\![x]\!] \cap [\![y]\!] = \emptyset$. Please note that x and y are fixed throughout this section.

The domain of $\mathcal{M}$ Let $\mathbb{T}$ be the set of all terms. We say that a set $X \subseteq \mathbb{T}$ is *proper* if it does not contain all terms.

We define the preorder $\leq$ on $\mathbb{T}$ as usual, setting $a \leq b$ if and only if $\Gamma \vdash \mathsf{all}\ a\ b$.

We say that a set X has the *xy-property* if for all sequences $\vec{r}$, if X contains both $\vec{r}\ \mathsf{only}\ x$ and $\vec{r}\ \mathsf{only}\ y$, then X also contains $\vec{r}\ \mathsf{only}\ z$ for all z. The xy-property is a strengthening of the requirement that a set not contain both x and y. Indeed, taking $\vec{r}$ to be the empty sequence, we see that no proper set with the xy-property contains both x and y. It may be regarded as an "induction loading device" in our construction below.

Finally, we define

$$M = \{X \subseteq \mathbb{T} : X \text{ is proper, up-closed, and has the } xy\text{-property}\}.$$

Some important sets of terms For any terms a and b, let $\{a, b\}^*$ be the set of all terms which lie on xy-sequences for Γ starting with a, b. If $\Gamma \vdash \mathsf{some}\ a\ b$, we would like to show that this is an element of M; it is the generic element of $[\![a]\!] \cap [\![b]\!]$.

In what follows, when we speak of xy-sequences, we mean xy-sequences from our fixed set Γ.

Note that if $s_1, \ldots, s_k$ and $t_1, \ldots, t_l$ are xy-sequences starting with a, b, then their concatenation $s_1, \ldots, s_k, t_1, \ldots, t_l$ is also an xy-sequence starting with a, b. So if $c, d \in \{a, b\}^*$, then there is some xy-sequence $t_1, \ldots, t_k$, starting with a, b, and containing both c and d.

Claim 6.7 *If Γ contains* $\mathsf{some}\ a\ b$*, then $\{a, b\}^*$ belongs to M.*

Proof First we check that $\{a, b\}^*$ is up-closed and has the xy-property. If $c \in \{a, b\}^*$ and $c \leq d$, then there is some xy-sequence $t_1, \ldots, t_k$ starting with a, b, with $c = t_i$ for some i. Then $t_1, \ldots, t_k, d$ is also an xy-sequence starting with a, b, so $d \in \{a, b\}^*$.

Similarly, if $\vec{r}\ \mathsf{only}\ x, \vec{r}\ \mathsf{only}\ y \in \{a, b\}^*$, then there is an xy-sequence $t_1, \ldots, t_k$, starting with a, b and containing both these terms. Then $t_1, \ldots, t_k, \vec{r}\ \mathsf{only}\ z$ is also an xy-sequence starting with a, b, for any term z.

It remains to show that $\{a, b\}^*$ is proper. If not, then $x, y \in \{a, b\}^*$, so there is some xy-sequence $t_1, \ldots, t_k$, starting with a, b, and containing both x and y.

We will use this sequence to build an instance of the scheme (R), such that Γ proves all its premises, from which we can conclude $\mathsf{some}\ x\ y$, contradicting our assumption in this proof.

Let φ_0 be $\mathsf{some}\ a\ b$. We have assumed $\Gamma \vdash \varphi_0$. For each $i \geq 3$, we define a sentence φ_i of the form $\mathsf{all}\ u_i\ v_i$, with $v_i = t_i$.

Case 1: There is $j < i$ such that $\Gamma \vdash \mathsf{all}\ t_j\ t_i$. Let φ_i be $\mathsf{all}\ t_j\ t_i$.

Case 2: There are $j, k < i$, a sequence $\vec{r}$, and a term z, such that t_j is $\vec{r}\ \mathsf{only}\ x$, t_k is $\vec{r}\ \mathsf{only}\ y$, and t_i is $\vec{r}\ \mathsf{only}\ z$. Let φ_i be $\mathsf{all}\ t_i\ t_i$, and note that $\Gamma \vdash \varphi_i$.

Now $\varphi_0, \varphi_3, \ldots, \varphi_k$ are the premises of an instance of the scheme (R) from which we can conclude $\mathsf{some}\ x\ y$. Indeed, the sentences have the appropriate form, so conditions 1 and 2 are satisfied. And x and y are in $\{a, b, v_3, \ldots, v_k\} = \{t_1, \ldots, t_k\}$, so condition 4 is satisfied. As for condition 3, if the sentence φ_i is introduced by Case 1, then $u_i = t_j$ for some $j < i$, i.e. u_i is in $\{a, b, v_3, \ldots, v_{i-1}\}$. If φ_i is introduced by Case 2, then $u_i = t_i$ is $\vec{r}\ \mathsf{only}\ z$, and $\vec{r}\ \mathsf{only}\ x$ and $\vec{r}\ \mathsf{only}\ y$ occur as t_j and t_k for some $j, k < i$, and hence are in $\{a, b, v_3, \ldots, v_{i-1}\}$. $\dashv$

Next, given an element $\mathcal{S} \in M$ and a binary atom $r \in \mathbf{R}$, we would like to define the generic element with the property that $\mathcal{S}[\![r]\!]\mathcal{S}_r$ (if there is any such element). Define

$$\mathcal{S}_r \quad = \quad \{x : r\ \mathsf{only}\ x \in \mathcal{S}\}.$$

Claim 6.8 *Let $\mathcal{S} \in M$ and $r \in \mathbf{R}$. Then $\mathcal{S}_r \in M$ is closed upwards and has the xy-property. Thus, if $\mathcal{S}_r$ is proper, it belongs to M.*

Proof If $v \leq w$ and $v \in \mathcal{S}_r$, then r only v belongs to $\mathcal{S}$. Using (MONO) from Section 4, we see that r only w belongs to $\mathcal{S}$. So $w \in \mathcal{S}_r$.

Suppose that $\mathcal{S}_r$ contains $\vec{s}$ only x and $\vec{s}$ only y. Then $\mathcal{S}$ contains r only ($\vec{s}$ only x) and r only ($\vec{s}$ only y). So for all z, $\mathcal{S}$ contains r only ($\vec{s}$ only z). And then $\mathcal{S}_r$ contains $\vec{s}$ only z as well. $\dashv$

The model Now we define our model $\mathcal{M}$ using M as its universe. The structure of $\mathcal{M}$ is again by Definition 4.3.

We check the Truth Lemma for $\mathcal{M}$: For all terms z,

$$[\![z]\!] \quad = \quad \{\mathcal{S} \in M : z \in \mathcal{S}\}.$$

The proof is by induction on z. For z atomic, this is by Definition 4.3. Assume our result for z, fix a binary atom r, and consider r only z. If r only z belongs to $\mathcal{S}$, then every point $\mathcal{T}$ such that $\mathcal{S}[\![r]\!]\mathcal{T}$ contains z. So by induction hypothesis, each such $\mathcal{T}$ belongs to $[\![z]\!]$. Thus $\mathcal{S} \in [\![r \text{ only } z]\!]$. In the other direction, suppose (r only z) $\notin \mathcal{S}$. Note that $\mathcal{S}_r$ is proper, since $z \notin \mathcal{S}_r$. So $\mathcal{S}_r$ belongs to M by our second claim above. By definition, $\mathcal{S}[\![r]\!]\mathcal{S}_r$. And as we have seen, $z \notin \mathcal{S}_r$. So by induction hypothesis, $\mathcal{S}_r \notin [\![z]\!]$. Thus $\mathcal{S} \notin [\![r \text{ only } z]\!]$. This completes the induction.

With the Truth Lemma proved, we verify that $\mathcal{M} \models \Gamma$. The all a b sentences in Γ are true in $\mathcal{M}$ due to the Truth Lemma and the fact that all elements of M are up-closed. For a sentence in Γ of the form some a b, we use the Truth Lemma and our first claim: $\{a, b\}^*$ belongs to M. Finally, we claim that $\mathcal{M} \not\models \mathsf{some}$ x y. For suppose that $\mathcal{S} \in [\![x]\!] \cap [\![y]\!]$. Then $x, y \in \mathcal{S}$. By the xy-property (with $\vec{r}$ the empty sequence), $\mathcal{S}$ contains all terms. But this contradicts the definition of M: every element of M is a proper set of terms.

This concludes our proof of Theorem 6.6.

7 Relational Languages with Complements

Up until now, we have studied languages with term constructors all or only, and also some. We get stronger logics by allowing complementation operators on both unary atoms (nouns) and binary atoms (transitive verbs). In general, we write the complement of either kind of atom with a "bar" overline: so if a is a unary atom, then $\overline{a}$ is its complement. We always assume that $\overline{\overline{a}} = a$ for all unary atoms, and the same for binary atoms.

The fragment $\mathcal{R}$ Formulas of $\mathcal{R}$:

$$\begin{array}{ll} \textsf{all}\ a\ c & \textsf{some}\ a\ \overline{c} \\ \textsf{all}\ a\ (r\ \textsf{all}\ b) & \textsf{some}\ a\ (\overline{r}\ \textsf{some}\ b) \\ \textsf{some}\ a\ (r\ \textsf{all}\ b) & \textsf{all}\ a\ (\overline{r}\ \textsf{some}\ b) \\ \textsf{all}\ a\ (r\ \textsf{some}\ b) & \textsf{some}\ a\ (\overline{r}\ \textsf{all}\ b) \\ \textsf{some}\ a\ (r\ \textsf{some}\ b) & \textsf{all}\ a\ (\overline{r}\ \textsf{all}\ b) \end{array} \tag{23}$$

Here a and b must be unary atoms (=nouns), and c must be either a unary atom or a complemented unary atom.

This complementation is a significant point. The main reason to have complemented *binary* atoms is to have a fragment which is closed under complementation. Indeed, we the listing above has each sentence φ paired with its complement $\overline{\varphi}$ on the same row. We need this on the first line, since c can be positive or negative. This serves a secondary purpose: we can gloss English sentences such as no a (r any b) by all a ($\overline{r}$ all b).

The fragment $\mathcal{R}^*$ This time we allow the subject nouns a in (23) to be of the forms r all c or r some c, where c is a unary atom. In other words, we require the nouns to be positive (as they are in $\mathcal{R}$), but we permit them to be complex nouns.

The fragment $\mathcal{R}^\dagger$ This fragment is defined as $\mathcal{R}$ is in (23), except that we permit a and b to be a complemented atoms (just as c is so permitted).

The fragment $\mathcal{R}^{*\dagger}$ This last fragment allows the subject nouns a in (23) to be of the form r all c or r some c, where c is a unary atom or a complemented unary atom; also, a and b might be complemented atoms.

Results on these fragments The main results on these fragments were established in [10]. They concern both logic and complexity. $\mathcal{R}$ has a purely syllogistic, finite proof system (see below), and its satisfiability problem is complete for NLogSpace. $\mathcal{R}^*$ has a syllogistic proof system with a rule of REDUCTIO AD ABSURDUM (RAA). It provably has no finite syllogistic system without (RAA). Its satisfiability problem is complete for NPTime. $\mathcal{R}^\dagger$ and $\mathcal{R}^{*\dagger}$ have no finite syllogistic systems even allowing (RAA). Their satisfiability problems are both complete for ExpTime.

7.1 Fragments with all, only, and Noun-Complements

At this point, we reformulate the fragments above using only instead of all.

The fragment $\mathcal{RO}$ Formulas of $\mathcal{RO}$:

$$\begin{array}{ll} \textsf{all } a\ c & \textsf{some } a\ \overline{c} \\ \textsf{all } a\ (r \textsf{ only } b) & \textsf{some } a\ (r \textsf{ some } \overline{b}) \\ \textsf{some } a\ (r \textsf{ only } b) & \textsf{all } a\ (r \textsf{ some } \overline{b}) \\ \textsf{all } a\ (r \textsf{ some } b) & \textsf{some } a\ (r \textsf{ only } \overline{b}) \\ \textsf{some } a\ (r \textsf{ some } b) & \textsf{all } a\ (r \textsf{ only } \overline{b}) \end{array} \tag{24}$$

Here a and b must both be unary atoms (=nouns), and c may be complemented. In contrast to what we saw in (23) there is no need to have complemented binary atoms. That is, negation may be defined in $\mathcal{RO}$ by going across the rows of (24).

The fragment $\mathcal{RO}^*$ This time we allow the subject nouns a in (24) to be of the forms r only c or r some d, where d is a unary atom. This is a direct parallel to the way we expanded $\mathcal{R}$ to get $\mathcal{R}^*$.

The fragment $\mathcal{RO}^\dagger$ As with the definition of $\mathcal{R}^\dagger$, the fragment $\mathcal{RO}^\dagger$ is defined as $\mathcal{RO}$ is in (24), except that we permit a and b to be complemented atoms.

The fragment $\mathcal{RO}^{*\dagger}$ This allows all atoms to be complemented, and the head nouns of sentences can be complex terms. For example, the fragment includes

$$\textsf{all } (\overline{s} \textsf{ only } \overline{a})\ (r \textsf{ only } \overline{b})$$

Observe that $\mathcal{R}^\dagger = \mathcal{RO}^\dagger$, and $\mathcal{R}^{*\dagger} = \mathcal{RO}^{*\dagger}$. This is by (3). It follows that the only new work to be done here would concern $\mathcal{RO}$ and $\mathcal{RO}^*$. At the time of this writing, the axiomatization and complexity results for these fragments are still open.

7.2 ExpTime Complexity of $\mathcal{RO}^\dagger$

We conclude this paper with a result on $\mathcal{RO}^\dagger$. The same result for $\mathcal{R}^\dagger$ was presented in Pratt-Hartmann and Moss [10], and our proof is really just a minor modification of the earlier argument. The main reasons for re-presenting it are that (1) the work here is slightly simpler; it does not go via a translation to first-order logic; (2) the direct use of the only logic might make the result more accessible to modal logicians (and for this reason we present the proof using modal notation).

Recall that a sentence in modal logic is satisfiable if it is true at some world in some model, and a set of sentences is satisfiable if there is some world in some model making all of them true.

Theorem 7.1 *[10] The problem of satisfiability problem for $\mathcal{RO}^\dagger$ is* EXPTIME-*complete.*

Proof For the upper bound, see Pratt-Hartmann [9], Theorem 3. So we are left with the lower bound.

The logic $\mathcal{L}(\Box, U)$ is the basic modal logic (with one modal operator) together with the modal operator U which we saw in (11). The satisfiability problem for $\mathcal{L}(\Box, U)$ is EXPTIME-hard. This result itself also goes back to Spaan [12] and is stated as Theorem 35 in Blackburn and van Benthem [2]. The proof may also be obtained as an easy adaptation of the corresponding result for propositional dynamic logic; see, e.g., Harel *et al.* [4]: 216 ff.

The overarching idea is to encode the satisfiability problem sentences of the logic $\mathcal{L}(\Box, U)$ into the satisfiability problem for sets of sentences in $\mathcal{RO}^\dagger$. It suffices, therefore, to reduce this problem to satisfiability in $\mathcal{RO}^\dagger$. Let φ be a formula of $\mathcal{L}(\Box, U)$.

We transform φ from $\mathcal{L}(\Box, U)$ into an equisatisfiable set of formulas in $\mathcal{RO}^\dagger$, but where we expand the basic vocabulary. Let

$$\begin{array}{lcl} \textsc{sub}(\varphi) & = & \text{all subformulas } \psi \text{ of } \varphi \\ \textsc{sub}_\neg(\varphi) & = & \textsc{sub}(\varphi) \cup \{\neg\psi : \psi \in \textsc{sub}(\varphi)\} \end{array}$$

Again, we take an input sentence φ and transform it into a set of sentences in an expanded language. This expanded language has all of the atomic sentences of φ, but it has more unary and binary atoms. We have a unary atom p_ψ for each $\psi \in \textsc{sub}_\neg(\varphi)$. When q is an atomic sentence of modal logic, we identify p_q with q itself. We also have a new unary atom o^*. We have an atom r for the relation underlying the $\Box$ and $\Diamond$ formulas of ordinary modal logic. We also have a new binary atom e; this is related to the universal modality of $\mathcal{L}(\Box, U)$. And for each conjunctive subformula $\psi \wedge \chi$ of φ, we have a binary atom $r_{\varphi\wedge\psi}$.

In what follows, we write $S+\varphi$ as a shorthand notation for $S \cup \{\varphi\}$. For $\psi \in \textsc{sub}(\varphi)$, define the set of $\mathcal{RO}^\dagger$-formulas T_ψ by recursion:

$$\begin{array}{rrcrr}
T_q = & \emptyset & & & \\
T_{\neg\psi} = & T_\psi & + & U(p_{\neg\psi} \to \neg p_\psi) + U(\neg p_\psi \to p_{\neg\psi}) & (25) \\
T_{\psi\wedge\chi} = & T_\psi \cup T_\chi & + & U(p_{\psi\wedge\chi} \to p_\psi) + U(p_{\psi\wedge\chi} \to p_\chi) & (26) \\
 & & + & U(\neg p_{\psi\wedge\chi} \to \Diamond_{\psi\wedge\chi} o^*) & (27) \\
 & & + & U(p_\psi \to \Box_{\psi\wedge\chi} p_\psi) + U(p_\chi \to \Box_{\psi\wedge\chi} \neg p_\psi) & (28) \\
T_{\Diamond\psi} = & T_\psi & + & U(p_{\Diamond\psi} \to \Diamond_r p_\psi) + U(\neg p_{\Diamond\psi} \to \Box_r \neg p_\psi) & \\
T_{U\psi} = & T_\psi & + & U(p_{U\psi} \to \Box_e p_{U\psi}) + U(p_{U\psi} \to \Box_r p_{U\psi}) & (29) \\
 & & + & U(p_{\neg U\psi} \to \Box_e p_{\neg U\psi}) + U(p_{\neg U\psi} \to \Box_r p_{\neg U\psi}) & (30) \\
 & & + & U(p_{U\psi} \to p_\psi) & (31) \\
 & & + & U(p_{\neg U\psi} \to \Diamond_e p_{\neg\psi}) & (32)
\end{array}$$

We check that φ is satisfiable if and only if $T_\varphi + \varphi$ is satisfiable.

Here is one direction. Let φ be satisfiable. Specifically, let $\mathcal{M}$ be a model, and suppose that the world w in it satisfies φ. Then we get a structure $\mathcal{M}^*$ for the larger language as follows:

$$\begin{array}{lcl}
M^* & = & \text{the universe } M \text{ of } \mathcal{M} \\
[\![p_\psi]\!] & = & \{x : x \models \psi\} \\
[\![o^*]\!] & = & M \\
[\![r]\!] & = & [\![r]\!] \text{ from } \mathcal{M} \\
[\![r_e]\!] & = & M \times M \\
[\![r_{\psi\wedge\chi}]\!] & = & \{(x,x) : x \not\models \psi \wedge \chi\}.
\end{array}$$

We see directly that all of the U sentences in the T-clauses above are true (at all points) in $\mathcal{M}^*$. The only interesting points are for the formulas in the sets $T_{\psi\wedge\chi}$, and the reasoning here is direct. (To verify (27) at a point x, use x itself. For (28), use that the only point which can be related to x by $[\![r_{\psi\wedge\chi}]\!]$ is x itself, and this would not hold if x did not satisfy $\varphi \wedge \psi$.) From this, an easy induction on ψ shows that every $w \in M$ satisfies T_ψ in $\mathcal{M}^*$. To conclude, recall that we have a world $w \in M$ such that $w \models \varphi$ in $\mathcal{M}$. This same w has $w \models T_\varphi + \varphi$ in $\mathcal{M}^*$.

Going the other way, suppose that we have a model $\mathcal{M}^*$ of the larger language, and suppose that w^* in it satisfies $T_\varphi + \varphi$. Let $\mathcal{M}$ be the model for standard modal logic defined as follows:

$$\begin{array}{rcl} M & = & \text{the points in } M^* \text{ reachable from } w^* \text{ in zero} \\ & & \text{or more steps using } [\![r]\!] \text{ and } [\![r_e]\!] \\ [\![q]\!] & = & \{x \in M : a \models q \text{ in } \mathcal{M}^*\} \\ [\![r]\!] & = & \{(x,y) \in M \times M : x[\![r]\!]y \text{ in } M^*\} \end{array}$$

Claim 7.2 *For all $a \in M$, $a \models \psi$ in $\mathcal{M}$ iff $a \models p_\psi$ in $\mathcal{M}^*$.*

Proof By induction on ψ. The base case of atomic sentences and the induction step for $\neg$ is trivial.

Here is the induction step for $\wedge$. Assume that $a \models p_{\psi\wedge\chi}$ in $\mathcal{M}^*$. By (26), $a \models p_\psi$ and also $a \models p_\chi$. By induction hypothesis, $a \models \psi$ and also $a \models \chi$. Thus $a \models \psi \wedge \chi$. In the other direction, suppose that $a \models \psi\wedge\chi$ in $\mathcal{M}$. Then in $\mathcal{M}^*$, a satisfies both p_ψ and p_χ. Suppose towards a contradiction that a does not satisfy $p_{\psi\wedge\chi}$. Then by (27), there is some b such that a is related to b by $[\![r_{\psi\wedge\chi}]\!]$. But then by (28), $b \models p_\psi$ and also $b \models p_{\neg\psi}$. But by induction hypothesis, we see that $b \models \psi$ and also $b \models \neg\psi$. This is a contradiction.

The induction step for $\Diamond$-formulas is quite close to that of $U\psi$, so we only will do the latter case in detail.

Let $a \in M$, and assume that $a \models p_{U\psi}$ in $\mathcal{M}^*$. Recall that a is reachable from w^*. If we had $w^* \models \neg p_{U\psi}$ in $\mathcal{M}^*$, then by (25) we have $w^* \models p_{\neg U\psi}$ in $\mathcal{M}^*$. By (30) and reachability, $a \models p_{\neg U\psi}$, and this would contradict (25). Thus $w^* \models p_{U\psi}$ in $\mathcal{M}^*$. Moreover, by (29), we see that for all $a' \in M$, $a' \models p_{U\psi}$ in $\mathcal{M}^*$. By (31), each $a' \in M$ satisfies p_ψ. By induction hypothesis, each $a' \in M$ satisfies ψ in $\mathcal{M}$. Thus $a \models U\psi$.

In the other direction, assume that $a \models \neg p_{U\psi}$ in $\mathcal{M}^*$. By (25), $a \models p_{\neg U\psi}$ in $\mathcal{M}^*$. By (32), there is some a' such that $[\![e]\!](a,a')$ and $a' \models p_{\neg\psi}$ in $\mathcal{M}^*$. This point a' is also reachable from w^* and hence belongs to M. By induction hypothesis, $a' \not\models \psi$ in $\mathcal{M}$. This shows that $a \models \neg U\varphi$ in $\mathcal{M}$.

This completes the proof of our claim. $\dashv$

Our claim done, we also have finished the proof of Theorem 7.1. $\dashv$

8 Conclusion

The main point of this paper has been to re-present the relational syllogistic using only instead of all. We have considered fragments with only which parallel what has been done with all. We have proved several completeness results. To re-state the open problems that we found along the way: find the efficient counter-model algorithm for $\mathcal{O}(\mathcal{RC})$; find the logics and complexities of the full logic of r only x and r some x; of $\mathcal{RO}$ and $\mathcal{RO}^{\dagger}$, It should be clear from the many problems that remain open that this are is far from settled.

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