RESEARCH STATEMENT

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What makes some problems easier to solve than others? What makes some areas of mathematics easier to understand than others? Much of mathematical logic is concerned with variants of these broad questions. Computer scientists classify problems by their computational complexity, reverse mathematicians measure non-constructive theorems by the strength of the comprehension and choice principles needed to prove them, and set theorists stratify axioms which go beyond ZFC set theory by their consistency strength.

Model theorists, on the other hand, look for dividing lines in the complexity of theories of mathematical structures in first-order logic, in an attempt to separate "tame" theories (e.g., the theory of vector spaces or algebraically closed fields) from "complex" ones (e.g., Peano arithmetic or ZFC set theory), and to develop general techniques for studying the tame theories and their models (a *model* for a theory T is just a mathematical structure satisfying the axioms of T). These tools allow us to gain an understanding of definability (sets, relations, and functions definable by first-order formulas) in tame structures (e.g. the ordered group of integers, the real field, or the complex field), which can be applied to purely mathematical questions about these structures. In the best case, we can also make analogies between different subfields or structures in mathematics precise, thus building bridges and porting techniques from one area to another.

Modern model theory is heavily influenced by Saharon Shelah's landmark work on classification theory [25]. Shelah developed a deep structure theory for the class of *stable* first-order theories and the models of these theories. Though very abstract, this work (and later developments in stability theory by many others working in the field) is a key component in many of the most striking applications of model theory, such as Ehud Hrushovki's proof of the Mordell–Lang conjecture for function fields [9].

The *neostability* program seeks to generalize Shelah's work to other dividing lines beyond stability. There are a host of these properties (see [22] for an online "map"), and unfortunatley they are often named by arcane acronyms like NSOP_1 or NTP_2. But as the theory develops, evidence accrues that some dividing lines are especially important and robust, demonstrated by structure theorems for theories on the "tame" side of the line and non-structure theorems for theories on the "complex" side. This program has been very successful for several classes of theories, most notably the simple theories (see [28]), the NIP theories (see [26]), and the o-minimal theories (see [27]).

In my work, I am particularly interested in *generic* and *random* structures, especially in relation to the neostability program described above. By generic, I usually mean *existentially closed* (though there are other important senses of "generic" and "random" which I will discuss later). Existential closedness is a richness condition on a model M which says that any first-order formula without quantifiers that has a solution in a larger model already has a solution in M. For example, a generic field is algebraically closed, and a generic linear order is dense without endpoints. When the generic models of a theory T can themselves be axiomatized by a first-order theory T^*, we call T^* the *model companion* of T.

The generic models of T are often more amenable to model-theoretic analysis than arbitrary models of T, and studying them can shed light on the entire class of models of T, especially when the model companion T^* exists (this is a generalization of the idea that it is useful in field theory to embed a field in its algebraic closure). Abraham Robinson already pioneered this approach to applied model theory in the 1950s, and it continues to bear fruit. For example, technical results about the stable theory DCF (the generic theory of fields equipped with a derivation) and the simple theory ACFA (the generic theory of fields equipped with a distinguished automorphism) have been central to the applications of model theory to arithmetic geometry alluded to above (e.g. [9] and [10]).

In the rest of this statement, I will describe three major themes in my current research:

(1) *Generic structures and NSOP$_1$.*
(2) *Fraïssé limits, random structures, and zero-one laws.*
(3) *Other logics (and connections with computer science).*

Generic structures and NSOP$_1$: One dividing line in the neostability hierarchy, called NSOP$_1$, has been the subject of increased attention recently. This is largely due to work of Artem Chernikov, Itay Kaplan, and Nick Ramsey [7, 12], who showed that NSOP$_1$ theories can be characterized by the existence of an abstract notion of independence called Kim-independence, which is a generalization of forking independence in simple and stable theories.

One of the most interesting features of NSOP$_1$ is its apparent robustness under "generic constructions". For example, in [17], Ramsey and I showed that the generic expansion of an NSOP$_1$ theory by Skolem functions, or by new constant, function, or relation symbols interpreted arbitrarily, is NSOP$_1$. A specific example is the generic theory of all $\mathcal{L}$-structures in an arbitrary language $\mathcal{L}$, which was previously known to be simple when $\mathcal{L}$ is relational. Using a result from Peter Winkler's thesis [29] on the existence of the generic Skolemization, it follows that any NSOP$_1$ theory which eliminates the quantifier $\exists^\infty$ has an expansion to an NSOP$_1$ theory with built-in Skolem functions.

In joint work with Gabe Conant [6], we showed that the theory of generic projective planes is NSOP$_1$ (a projective plane is an incidence structures in which any two points are incident with a unique line and any two lines are incident with a unique point), and we generalized this to structures we call (m, n)-pseudoplanes (incidence structures in which any m points are incident with $(n-1)$ lines and any n lines are incident with $(m-1)$ points). These examples are particularly interesting, because viewed as generic bipartite graphs omitting $K_{m,n}$, they are bipartite analogues of the generic K_n-free graphs, which are the canonical examples of SOP$_3$ but NSOP$_4$ theories.

During my postdoc at Indiana University, I continued this work with an undergraduate student, Matisse Peppet, in a successful summer REU project. Matisse investigated the combinatorics of (m, n)-pseudoplanes, identified a robust notion of non-degeneracy for these structures, and showed that if $n \geq 2$ and $m \geq 3$, then every non-degenerate (m, n)-psuedoplane is infinite. In particular, her work implies that in these cases, the theory of generic (m, n)-pseudoplanes has no prime model. The corresponding statement for projective planes (the case $n = m = 2$) is equivalent to an open problem of Erdős.

The lines of work described above led to a fruitful collaboration with Chieu-Minh Tran and Erik Walsberg, on a construction we call interpolative fusions (which generalizes another part of Winkler's thesis [29]). Given a family of languages $(\mathcal{L}_i)_{i\in I}$ with common intersection $\mathcal{L}_\cap$ and a family $(T_i)_{i\in I}$ of model-complete $\mathcal{L}_i$-theories, with a common set $T_\cap$ of $\mathcal{L}_\cap$-consequences, the interpolative fusion is the model companion of the union $\bigcup_{i\in I} T_i$ (if it exists). Interpolative fusions provide a unified framework for studying a wide variety of examples of generic theories in model theory, some of which (e.g. algebraically closed fields with multiple independent valuations) are explicitly interpolative fusions, while others (e.g. DCF and ACFA) are bi-interpretable with interpolative fusions.

In my first paper with Tran and Walsberg [19], we provided sufficient conditions for the existence of the interpolative fusion, generalizing many existence results for model companions in the literature. In our second paper [20], we proved quantifier-elimination results and showed that under appropriate hypotheses on $T_\cap$ (including stability), if all T_i are NSOP_1, then the interpolative fusion is NSOP_1. This gives an extremely flexible recipe for producing new NSOP_1 theories, which can be used to motivate and test conjectures in this area.

Many open problems remain in the area of interpolative fusions. One of the most interesting, which is current work in progress, is under what conditions imaginaries (i.e., quotients by definable equivalence relations) in the interpolative fusion can be reduced to imaginaries in the theories T_i.

The phenomenon that generic constructions tend to produce NSOP_1 theories suggests a broad problem in neostability: is there a class of theories which is as robust as NSOP_1 under generic constructions, but which also contains ordered structures? One motivation comes from NTP_2, a class of theories containing the simple theories, but also some theories with the strict order property, such as dense linear orders and p-adic fields.

Problem 1. Complete the analogy: simplicity is to NTP_2 as NSOP_1 is to X. Develop a theory of Kim-independence in the context of theories with property X.

The class of theories with the conjectural property X should include the NTP_2 theories and the NSOP_1 theories, and thus a solution to this problem would provide the broadest class of theories for which a satisfying theory of independence has been developed to date.

Toward this end, Ramsey and I introduced in [18] a property called New Kim's Lemma, as well as a syntactic property called NBTP. We proved that NBTP theories satisfy New Kim's Lemma, which is a promising step toward developing a theory of independence in this regime. It is an open problem to determine whether NBTP and New Kim's Lemma are equivalent.

We also verified New Kim's Lemma for a number of natural examples of first-order theories which should satisfy property X, but which are not on the tame side of any other known model-theoretic dividing lines: e.g. the generic theory of parameterized linear orders, and the theory of real closed fields with a symmetric positive-definite (or, alternatively, an alternating and non-degenerate) bilinear form.

New Kim's Lemma and NBTP are two possible solutions to Problem 1, but other possible solutions exist in the literature, such as the property NATP [2] (the equivalence of NATP and NBTP is also open). There is much more work to be done to determine whether our definitions provide the robust dividing line that we seek.

Fraïssé limits, Random structures, and zero-one laws: Another sense of the word generic refers to Fraïssé limits. If $\mathcal{K}$ is a class of finite structures (think of the class of finite graphs) satisfying certain hypotheses, then there is a countably infinite structure $M_\mathcal{K}$, the Fraïssé limit of $\mathcal{K}$, which is universal (every structure in $\mathcal{K}$ embeds into it) and homogeneous (any two such embeddings are conjugate by an automorphism of $M_\mathcal{K}$). If we look at the space $X_\mathcal{K}$ of all structures M with domain $\mathbb{N}$, such that all finite substructures of M are in $\mathcal{K}$, then the isomorphism class of the Fraïssé limit $M_\mathcal{K}$ is *comeager* in $X_\mathcal{K}$: generic from the point of view of view of Baire category. Fraïssé limits are at the heart of many connections between model theory and combinatorics, descriptive set theory, and permutation group theory (see [4] and [13]), and they show up frequently in my work.

If $\mathcal{G}$ is the class of the class of finite graphs, the Fraïssé limit $M_{\mathcal{G}}$ is known as the *Rado graph*, or the *random graph*. The Rado graph also arises from the Erdős–Rényi random graph construction: fix a countably infinite vertex set and a probability $0 < p < 1$, and put an edge between each pair of distinct vertices independently with probability p. The resulting graph is isomorphic to the Rado graph with probability 1.

This random construction can be formalized as a probability measure μ on the space $X_{\mathcal{G}}$, which is moreover invariant and ergodic for the natural group action of S_∞ on this space (this group action is sometimes called the "logic action", and its orbits are exactly the isomorphism classes). It turns out that this kind of measure on $X_{\mathcal{G}}$, which we call an *ergodic structure*, encodes exactly the same information as a *graphon*, a limit structure for a sequence of finite graphs which converges in the appropriate sense (see [21]), and its natural generalization to other spaces $X_{\mathcal{K}}$ can be viewed as "generalized graphons" for other classes of structures $\mathcal{K}$.

The Baire category / measure analogy between Fraïssé limits and ergodic structures was a major theme of my PhD thesis [15]. In a joint paper [1] with Nate Ackerman, Cameron Freer, and Rehana Patel that came out of that thesis, we characterized those ergodic structures which (unlike the Erdős–Rényi measure) do not give measure 1 to any single isomorphism class. The proof used a detailed model-theoretic "Morley–Scott analysis" of ergodic structures, providing evidence that these measures can be profitably viewed as *random* analogues of ordinary structures.

In addition to being the Fraïssé limit of the class $\mathcal{G}$ of finite graphs and arising (with probability 1) from the Erdős–Rényi random construction, the Rado graph can be characterized in a third way, via the zero-one law for finite graphs. For every first-order sentence φ in the language of graphs, the limit of the fraction of graphs of size n satisfying φ is 0 or 1 as $n \to \infty$. And the Rado graph is the unique countable model of the set of all sentences with limiting probability 1. In particular, it is pseudofinite: every sentence in its theory has a finite model.

A major open problem in combinatorial model theory is the question of whether the generic triangle-free graph (which has SOP_3, hence is not simple or NSOP_1) is pseudofinite. In [14] (which is a chapter of my PhD thesis [15]), I was able to show pseudofiniteness for several Fraïssé limits with NSOP_1 theories. This led me to raise the following question, which seems very difficult.

Question 2. For every pseudofinite Fraïssé limit $M_{\mathcal{K}}$, is $\mathrm{Th}(M_{\mathcal{K}})$ NSOP_1?

In work in progress with my Wesleyan colleague Cameron Hill, we are working on more tractable versions of the open problems above. In [14], I also showed that if a Fraïssé class $\mathcal{K}$ has n-DAP for all n (disjoint n-amalgamation, a "higher dimensional" analogue of the amalgamation property), then there is a sequence of measures $(\mu_k)_{k\in\mathbb{N}}$ on the space of structures in $\mathcal{K}$ with domain $[k]$, which converge (in the appropriate sense) to an ergodic structure μ, such that μ gives measure 1 to the isomorphism class of the Fraïssé limit $M_{\mathcal{K}}$, and the μ_k have a zero-one law converging to the theory of $M_{\mathcal{K}}$. If $\mathcal{K}$ admits such a sequence of measures, I call the theory $\mathrm{Th}(M_{\mathcal{K}})$ *strongly pseudofinite*. In the case of the class $\mathcal{G}$ of finite graphs, the uniform measures on the spaces of graphs with domain $[k]$ witness that the theory of the Rado graph is strongly pseudofinite.

Conjecture 3. Let $\mathcal{K}$ be a Fraïssé class such that $\mathrm{Th}(M_{\mathcal{K}})$ is strongly pseudofinite. Then $\mathrm{Th}(M_{\mathcal{K}})$ is simple (with SU-rank 1).

To prove the conjecture, we seek to show that if $\mathrm{Th}(M_{\mathcal{K}})$ is strongly pseudofinite, then $\mathcal{K}$ is close (in a precise sense) to having n-DAP. A solution to the conjecture would establish that the generic triangle free graph is not strongly pseudofinite.

In current work in progress with Artem Chernikov and Alex Van Abel, we are investigating another strengthening of pseudofiniteness, introduced recently by Chernikov: A complete theory T with infinite models is *faux finite* if there is a sentence $\psi \in T$ such that for every sentence φ, $\varphi \in T$ if and only if φ is true in *all* sufficiently large finite models of ψ. This notion is closely related to quasi-finite axiomatizability, but defines a broader class of theories than just the pseudofinite quasi-finitely axiomatizable ones.

In a recent paper with Isaac Goldbring and Bradd Hart [8], we established a zero-one law for the class $\mathcal{K}$ of finite metric spaces of diameter ≤ 1 in continuous logic, an extension of first-order logic in which sentences take truth values in the unit interval $[0, 1]$. Although the class $\mathcal{K}$ has a Fraïssé limit, called the Urysohn sphere, we found that almost-sure theory (the set of sentences with limiting probability 1) differs from the theory of the Urysohn sphere; in fact, the almost-sure theory asserts that all non-zero distances are in the interval $[\frac{1}{2}, 1]$. This was the first result in the literature on zero-one laws in continuous logic, and many interesting questions remain for future work. To me, the most interesting is the question of whether the Urysohn sphere is pseudofinite (in the sense of continuous logic).

Other logics (and connections with computer science): I am also interested in non-first-order logics, categorical logic, and connections with computer science.

First, my two papers with my postdoc mentor Larry Moss ([23] and [16]) at Indiana University are part of his long-term project on natural logics: logics with features which model natural language. These logics are typically much weaker than first-order logic; they are decidable, and sometimes even computationally tractable. To understand these logics, one needs to prove precise enough completeness theorems to determine the computational complexity of the consequence relation $\models$. We analyzed a particular family of natural logics and identified features which distinguish tractable logics ($\models$ is in P) from intractable logics ($\models$ is co-NP hard). These dichotomies have implications for automated reasoning from natural language.

In joint work with Siddharth Bhaskar, also during my postdoc at IU, we studied the neostability dividing lines from model theory in the context of least fixed point (inductive) logic over classes of finite structures [3]. In particular, we showed that in this setting, stability and NIP coincide. The study of inductive logic is motivated by notions of computation over finite data structures and is related to issues in finite model theory.

Finally, in current work, I am developing a categorical extension of first-order logic that replaces the variable contexts of formulas and the underlying sets of structures with objects in categories with sufficient structure, e.g. locally finitely presentable categories. By dualizing, I obtain a "cologic", which naturally extends semantics, notions, and methods of model theory to profinite structures and coalgebras. This unifies some earlier examples of model-theoretic methods for profinite structures in the literature (see [5] and [11]). And it promises to provide logical tools for working coalgebras, which are frequently used in computer science to model infinite data types (see [24]). I have a paper in preparation laying out the foundations, but there is much work to be done here developing the model theory.

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