Interpolative Fusions

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Outline

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1. What are interpolative fusions?
2. What are some examples?
3. When do they exist, and how can we axiomatize them?
4. How much quantifier elimination do they have?
5. What about neostability? (stability, NIP, simplicity, $NSOP_1$, etc.)

This is all joint work with Erik Walsberg and Minh Chieu Tran.

- *Interpolative fusions I*
    - Covers questions (1)-(4).
    - Forthcoming: soon!
- *Interpolative fusions II*
    - Covers question (5).
    - Forthcoming: not as soon!

Interpolative structures

Suppose we have:

- A language $L_\cap$.
- A family of languages $(L_i)_{i\in I}$ with $L_i \cap L_j = L_\cap$ for $i \neq j$.
- $L_\cup = \bigcup_{i\in I} L_i$.
- $\mathcal{M}_\cup$ an $L_\cup$-structure with reducts $\mathcal{M}_i$ to L_i and $\mathcal{M}_\cap$ to $L_\cap$.

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We say $\mathcal{M}_\cup$ is an **interpolative structure** if for all families $(X_i)_{i\in J}$ such that $J \subseteq I$ is finite and each X_i is an $\mathcal{M}_i$-definable set, either:

1. $\bigcap_{i\in J} X_i \neq \emptyset$, or
2. There is a family $(Y^i)_{i\in J}$ of $\mathcal{M}_\cap$-definable sets such that

$$X_i \subseteq Y^i \text{ for all } i \in J, \text{and } \bigcap_{i\in J} Y^i = \emptyset.$$

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Idea: The structures $\mathcal{M}_i$ interact "randomly" / "generically" subject to the constraints imposed by their common reduct $\mathcal{M}_\cap$.

Interpolative fusions

Now suppose we have:

- An $L_\cap$-theory $T_\cap$.
- An L_i-theory T_i for each $i \in I$, such that $T_\cap$ is the set of $L_\cap$-consequences of T_i.
- $T_\cup = \bigcup_{i \in I} T_i$.

If the class of interpolative models of $T_\cup$ is elementary, we call the theory $T^*_\cup$ of this class the **interpolative fusion** of $(T_i)_{i \in I}$ over $T_\cap$.

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Proposition

If each T_i is model-complete, then:

1. $\mathcal{M}_\cup \models T_\cup$ *is interpolative if and only if it is existentially closed among models of* $T_\cup$.
2. $T_\cup^*$ *is the model companion of* $T_\cup$ *(if it exists).*

The main component of the proof is the Craig interpolation theorem.

Examples: Minh's theory

Let $\chi\colon \overline{\mathbb{F}_p}^\times \to \mathbb{C}^\times$ be an injective multiplicative character.

The image of χ is contained in the unit circle in $\mathbb{C}$, so it induces a circular order C_χ on $\overline{\mathbb{F}_p}^\times$.

Theorem (Tran)

$\mathrm{Th}(\overline{\mathbb{F}_p}, 0, 1, +, -, \times, C_\chi)$ *is (a completion of) the interpolative fusion of* $\mathrm{Th}(\overline{\mathbb{F}_p}, 0, 1, +, -, \times)$ *and* $\mathrm{Th}(\overline{\mathbb{F}_p}, 0, 1, \times, C_\chi)$ *over* $\mathrm{Th}(\overline{\mathbb{F}_p}, 0, 1, \times)$.

The proof uses the Lang-Weil estimates (and other black boxes).

$+$ and C_χ interact "randomly" / "generically" over $\times$.

Examples: Winkler's thesis

Theorem (Winkler)

Let T_1 and T_2 be theories in disjoint languages L_1 and L_2. If T_1 and T_2 are model complete and eliminate $\exists^\infty$, then $T_1 \cup T_2$ has a model companion.

The model companion is the interpolative fusion of T_1 and T_2 over $T_\cap$, where $T_\cap$ is the theory of an infinite set in the empty language $L_\cap$.

Special case: When $T_2 = T_\cap$ is the theory of an infinite L_2-structure, then the interpolative fusion $T^*_\cup$ is the **generic expansion** of T_1 to $L_1 \cup L_2$.

Examples: Fields with multiple structures

The generic theory of fields with n independent valuations is the interpolative fusion of n copies of ACFV over ACF.

More generally, you can put together copies of your favorite structures on fields (valuations, derivations, automorphisms, etc.) over ACF – when the interpolative fusion exists.

Examples: Fields with multiple structures

Generic theories of fields with several independent valuations were first studied by Lou van den Dries in his thesis. Here is a quote from that thesis:

> "P. Winkler treats in [Wi] some general constructions on model complete theories giving, under certain conditions, new model complete theories. For instance, he proves that the disjoint union of two theories each having an algebraically bounded model companion has a model companion. Now in our case not a disjoint union of theories is considered, but what might call, an amalgamated union, with the theory of domains as common part. It seems to me that something like algebraic boundedness is really behind the proof of (1.6). All this suggests a common generalization of Winkler's an my results."

(1.6) is the existence of the model companion for theories of fields with several orderings and valuations.

Examples: ACFA and T_A

ACFA is not an interpolative fusion, but it is bi-interpretable with one.

1. $T_\cap$ is the two-sorted theory of two algebraically closed fields K and K' of the same characteristic (with no connection between them).
2. T_1 is the expansion of $T_\cap$ by an isomorphism $\sigma_1\colon K \to K'$.
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Both T_1 and T_2 are bi-interpretable with ACF, and $T_\cup$ is bi-interpretable with the theory ACF_σ of an algebraically closed field equipped with an automorphism σ. (Take $\sigma = (\sigma_2)^{-1} \circ \sigma_1$).

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It follows that it restricts to a bi-interpretation between the interpolative fusion $T_\cup^*$ and the model companion ACFA of ACF_σ.

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For arbitrary T, the theory T_A of a model of T with a generic automorphism is bi-interpretable with an interpolative fusion when it exists.

Examples: DCF and fields with operators

DCF is not an interpolative fusion, but it is bi-interpretable with one.

Let k be any ring.

- Let $D(k) = k[\varepsilon]/(\varepsilon^2)$.
- Define $\pi\colon D(k) \to k$ by $\pi(a + b\varepsilon) = a$.

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- *Viewing $D(k)$ as an abstract ring equipped with π and ε:*
  - One section ρ_1 of π gives $D(k)$ a k-algebra structure.
  - A second section ρ_2 of π defines a derivation on k.

Examples: DCF and fields with operators

Let $L_\cup$ be the language with:

- Sorts k and D, with the language of rings on each sort.
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1. For $i = 1, 2$, T_i is the theory of $(k, D(k), \pi, \varepsilon, \rho_i)$, where k is algebraically closed and $\rho_i\colon k \to D(k)$ is the standard k-algebra structure on $D(k)$. This is bi-interpretable with ACF.
2. $T_\cap$ is the common reduct of T_1 and T_2 which forgets ρ_1 and ρ_2.
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More generally, any generic theory of $\mathcal{D}$-fields (fields with operators) in the sense of Moosa and Scanlon is bi-interpretable with an interpolative fusion.

The main questions

Axiomatization results:
When does $T^*_\cup$ exist? I.e., when is the class of interpolative models of $T_\cup$ elementary?

Preservation results:
How can we understand properties of $T^*_\cup$ in terms of properties of the T_i and $T_\cap$?

We seek to generalize results and proofs about individual examples, placing them in the general framework of interpolative fusions.

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The pseudo-topological setting

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Idea: If all the X_i are "dense" in the same $\mathcal{M}_\cap$-definable set, they can't be separated by $\mathcal{M}_\cap$-definable sets.

The pseudo-topological setting

Let $\mathbb{M} \models T$, and let dim assign an ordinal or the formal symbol $-\infty$ to each $\mathbb{M}$-definable set, such that for all $\mathbb{M}$-definable X, X':

1. $\dim(X \cup X') = \max\{\dim X, \dim X'\}$,
2. $\dim X = -\infty$ if and only if $X = \emptyset$,
3. $\dim X = 0$ if and only if X is nonempty and finite,

We call such $\dim$ an **ordinal rank** on T.

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Let X be a definable set and A be an arbitrary set.

- A is **pseudo-dense** in X if A intersects every non-empty definable $X' \subseteq X$ such that $\dim X' = \dim X$.
- X is a **pseudo-closure** of A if $A \subseteq X$ and A is pseudo-dense in X. (Note the pseudo-closure is not unique, in general.)

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Let $\mathbb{M}'$ be an expansion of $\mathbb{M}$. Then $\mathbb{M}'$ is **approximable** if every $\mathbb{M}'$-definable set admits an $\mathbb{M}$-definable pseudo-closure.
We also say $T' = \mathrm{Th}(\mathbb{M}')$ is **approximable** over T.

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T' **defines pseudo-density** over T if for all L-formulas $\varphi(x, y)$ and L'-formulas $\psi(x, z)$ there is an L'-formula $\delta'(y, z)$ such that $\psi(\mathbb{M}', c)$ is pseudo-dense in $\varphi(\mathbb{M}', b)$ if and only if $\mathbb{M}' \models \delta(b, c)$.

Theorem

If $T_\cap$ admits an ordinal rank, and each T_i is approximable over $T_\cap$ and defines pseudo-density over $T_\cap$, then $T^*_\cup$ exists.

This theorem also has a "relativized" version, in which the definability of pseudo-density only needs to be checked on a sufficiently rich collection of "pseudo-cells".

Consequences

The content of the previous theorem can be elaborated in different ways in various contexts. Here are two sample applications:

When T is o-minimal, any expansion defines pseudo-density over T.

Theorem

Suppose $T_\cap$ is o-minimal. If $T_\cap$ is an open core of each T_i (the topological closure of every $\mathcal{M}_i$-definable set is $\mathcal{M}_\cap$-definable), then $T^*_\cup$ exists.

When T is ω-stable, any expansion is approximable over T.

Theorem

Suppose that $T_\cap$ is ω-stable and ω-categorical with weak e.i. If each T_i eliminates $\exists^\infty$, then $T^*_\cup$ exists.

The main questions

Axiomatization results:
When does $T_\cup^*$ exist? I.e., when is the class of interpolative models of $T_\cup$ elementary?

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From now on, we assume:

- $T_\cup^*$ **exists.**
- **Each** T_i **has quantifier elimination.**

acl-completeness

A theory T is acl-**complete** if for all $\mathcal{M} \models T$, and $A = \mathrm{acl}(A) \subseteq \mathcal{M}$, every embedding $f\colon A \to \mathcal{N} \models T$ is partial elementary.

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The **combined closure**, $\mathrm{ccl}(A)$, of a subset A of $\mathcal{M}_\cup \models T^*_\cup$ is the smallest set containing A which is acl_i-closed for each $i \in I$:

$$b \in \mathrm{ccl}(A) \iff b \in \mathrm{acl}_{i_n}(\dots(\mathrm{acl}_{i_1}(A))\dots) \text{ for some } i_1, \dots, i_n \in I.$$

(Here acl_i is acl in the reduct to L_i.)

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(Here acl_i is acl in the reduct to L_i.)

Theorem

Suppose $T_\cap$ is stable with weak e.i. Then $\mathrm{acl}_\cup = \mathrm{ccl}$ and $T_\cup^*$ is acl-complete.

So if $A = \mathrm{ccl}(A)$, then

$$T_\cup^* \cup \bigcup_{i \in I} \mathrm{tp}_{L_i}(A) \models \mathrm{tp}_{L_\cup}(A).$$

acl-completeness

The key tool in the proof is the following lemma.

Let $L \subseteq L'$, and let T' be an L'-theory with L-reduct T. Assume T is stable with weak e.i., and write $\mathop{\smile\!\!\!\!\!\!|}^r$ for forking independence in the reduct.

Lemma (Full existence over $\mathrm{acl}_{L'}$-closed sets)

For any $C = \mathrm{acl}_{L'}(C)$ and any B, there exists A^* with $A^* \mathop{\smile\!\!\!\!\!\!|}^r_C B$ and $\mathrm{tp}_{L'}(A^*/C) = \mathrm{tp}_{L'}(A/C)$.

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Remarks:

1. For the lemma, it suffices to assume T is simple with stable forking and geometric e.i. But we also need stationarity to prove acl-completeness.
2. The hypothesis "$T_\cap$ is stable with weak e.i." can be replaced here (and in what follows) by the existence of an independence relation satisfying full existence and stationarity over ccl-closed sets.

Quantifier elimination, stability, NIP

Theorem

Suppose $T_\cap$ is stable with weak e.i. and $\mathrm{acl}_i(A) = \mathrm{dcl}_\cap(A)$ for all sets A and all $i \in I$. Then every $L_\cup$-formula is $T^*_\cup$-equivalent to a Boolean combination of quantifier-free L_i-formulas.

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Corollary (Same hypotheses)

If each T_i is stable/NIP, then $T^*_\cup$ is stable/NIP.

- *Proof:* preservation of stability/NIP under Boolean combinations.
- Under these hypotheses, κ-stability is also preserved in interpolative fusions. (*Proof:* Type counting.)
- Slightly weaker (but more technical) hypotheses suffice. But we don't hope to get QE except under tight control on acl.

TP_2

Interpolative fusions can have TP_2, even when fusing two ω-stable theories in disjoint languages.

Example:

- $T_\cap$ is the theory of an infinite set in the empty language.
- T_1 is the theory of divisible abelian groups in the language $\{0, +, -\}$.
- T_2 is the theory of an equivalence relation with infinitely many infinite classes in the language $\{E\}$.

$$\varphi(x; y, z) : (x + y)Ez \text{ has } \mathsf{TP}_2 \text{ in } T^*_\cup.$$

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Let $(v_i)_{i\in\omega}$ be distinct, let $(e_j)_{j\in\omega}$ be representatives of distinct equivalence classes, and set $a_{i,j} = (v_i, e_j)$.

- $\{(x + v_n)Ee_{\sigma(n)} \mid n < \omega\}$ is consistent, while
- $\{(x + v_n)Ee_i, (x + v_n)Ee_j\}$ is inconsistent when $i \neq j$.

Preservation of NSOP$_1$

So it's natural to ask: what about NSOP$_1$?

Theorem (Kaplan–Ramsey)

T is NSOP$_1$ if and only if there is a relation $\mathop{\smile\!\!\!\!\!\!|}\,_M$ (Kim independence) defined on subsets of the monster model $\mathbb{M}$ for all $M \prec \mathbb{M}$ such that:

1. *Invariance: If $A \mathop{\smile\!\!\!\!\!\!|}\,_M B$ and $A'B'M' \equiv ABM$, then $A' \mathop{\smile\!\!\!\!\!\!|}\,_M B'$.*
2. *Symmetry: If $A \mathop{\smile\!\!\!\!\!\!|}\,_M B$, then $B \mathop{\smile\!\!\!\!\!\!|}\,_M A$.*
3. *Monotonicity: If $A' \subseteq A$, $B' \subseteq B$, and $A \mathop{\smile\!\!\!\!\!\!|}\,_M B$, then $A' \mathop{\smile\!\!\!\!\!\!|}\,_M B'$.*
4. *Existence: $A \mathop{\smile\!\!\!\!\!\!|}\,_M M$.*
5. *Strong finite character: if $A \not\mathop{\smile\!\!\!\!\!\!|}\,_M B$, then there is a formula $\varphi(x;b) \in \mathrm{tp}(A/MB)$ such that for any $a' \models \varphi(x;b)$, $a' \not\mathop{\smile\!\!\!\!\!\!|}\,_M b$.*
6. **The independence theorem:** *If $a \mathop{\smile\!\!\!\!\!\!|}\,_M B$, $a' \mathop{\smile\!\!\!\!\!\!|}\,_M C$, $B \mathop{\smile\!\!\!\!\!\!|}\,_M C$, and $a \equiv_M a'$, then there exists a'' such that $a'' \equiv_{MB} a$, $a'' \equiv_{MC} a$, and $a'' \mathop{\smile\!\!\!\!\!\!|}\,^K_M BC$.*

Preservation of NSOP$_1$

Theorem

Suppose $T_\cap$ is stable with weak e.i. and 3-uniqueness. If each T_i is NSOP$_1$, then $T_\cup^*$ is NSOP$_1$.

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Suppose $T_\cap$ is stable with weak e.i. and 3-uniqueness. If each T_i is NSOP$_1$, then $T_\cup^*$ is NSOP$_1$.

Definition

Suppose a_1, a_2, and a_3 enumerate algebraically closed sets, pairwise $\mathop{\smile\!\!\!\!\!\!|}^f$-independent over a common algebraically closed subset A. For $1 \leq i < j \leq 3$, let a_{ij} be a tuple enumerating $\mathrm{acl}(a_i, a_j)$. T has 3-*uniqueness* if $\mathrm{tp}(a_{12}) \cup \mathrm{tp}(a_{13}) \cup \mathrm{tp}(a_{23}) \vdash \mathrm{tp}(a_{12}a_{13}a_{23})$.

To get amalgamation (acl-completeness) in $T_\cup^*$, we assumed weak elimination of imaginaries $\implies$ stationarity $=$ "2-uniqueness" in $T_\cap$.

To get 3-amalgamation in T (the independence theorem), we assume 3-uniqueness $=$ elimination of "generalized imaginaries" (groupoids).

Proof sketch

Define $A \mathop{\smile\!\!\!\!\!\!\;|}_M B \iff \mathrm{ccl}(MA) \mathop{\smile\!\!\!\!\!\!\;|}^K_M \mathrm{ccl}(MB)$ in each reduct L_i.
To prove the independence theorem:

- Given a, a', A, B, separately apply the independence theorem in each reduct, obtaining an a'' in each reduct.
- All these amalgams are guaranteed to agree on $\mathrm{tp}_{L_\cap}(\mathrm{acl}_\cap(a''AB))$ by 3-uniqueness.
- To handle the elements which are in ccl but not $\mathrm{acl}_\cap$, we need a stronger form of the independence theorem which implies that we can take $\mathrm{ccl}(a''A) \mathop{\smile\!\!\!\!\!\!\;|}^f_{\mathrm{acl}_\cap(a''AB)} \mathrm{ccl}(a''B)$, $\mathrm{ccl}(a''A) \mathop{\smile\!\!\!\!\!\!\;|}^f_{\mathrm{acl}_\cap(a''AB)} \mathrm{ccl}(AB)$, and $\mathrm{ccl}(a''B) \mathop{\smile\!\!\!\!\!\!\;|}^f_{\mathrm{acl}_\cap(a''AB)} \mathrm{ccl}(AB)$ in $\mathcal{L}_\cap$.
- Then we can apply 3-uniqueness again, over $\mathrm{acl}_\cap(a''AB)$ this time. This implies that the two amalgams agree on all of $\mathrm{ccl}(a''AB)$.
- Finally, apply the Robinson Joint Consistency Theorem.

The "reasonable" independence theorem

Let T_1 be an NSOP$_1$ theory with a reduct T_0 which is simple with stable forking and geometric elimination of imaginaries.

$$\text{Define } A \mathop{\smile\!\!\!\!\!\!|}^{r}_{C} B \iff \mathrm{acl}_1(AC) \mathop{\smile\!\!\!\!\!\!|}^{f}_{\mathrm{acl}_1(C)} \mathrm{acl}_1(BC) \text{ in } \mathbb{M}_0.$$

where acl_1 is algebraic closure in $\mathbb{M}_1$.

Example: If T_0 is the theory of an infinite set, then $\mathop{\smile\!\!\!\!\!\!|}^{r} = \mathop{\smile\!\!\!\!\!\!|}^{a}$.

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Theorem (Independence theorem)

If $a \mathop{\smile\!\!\!\!\!\!|}^{K}_{M} b$, $a' \mathop{\smile\!\!\!\!\!\!|}^{K}_{M} c$, $b \mathop{\smile\!\!\!\!\!\!|}^{K}_{M} c$, and $a \equiv_M a'$, then there exists a'' such that $a'' \equiv_{Mb} a$, $a'' \equiv_{Mc} a$, and $a'' \mathop{\smile\!\!\!\!\!\!|}^{K}_{M} bc$.

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Example: If T_0 is the theory of an infinite set, then $\mathop{\smile\!\!\!\!\!\!\mid}^{r} = \mathop{\smile\!\!\!\!\!\!\mid}^{a}$.

Theorem (K., K.–Ramsey in the case $\mathop{\smile\!\!\!\!\!\!\mid}^{r} = \mathop{\smile\!\!\!\!\!\!\mid}^{a}$)

If $a \mathop{\smile\!\!\!\!\!\!\mid}^{K}_{M} b$, $a' \mathop{\smile\!\!\!\!\!\!\mid}^{K}_{M} c$, $b \mathop{\smile\!\!\!\!\!\!\mid}^{K}_{M} c$, and $a \equiv_M a'$, then there exists a'' such that $a'' \equiv_{Mb} a$, $a'' \equiv_{Mc} a$, and $a'' \mathop{\smile\!\!\!\!\!\!\mid}^{K}_{M} bc$, **and further,**

$$a'' \mathop{\smile\!\!\!\!\!\!\mid}^{r}_{Mb} c, \quad a'' \mathop{\smile\!\!\!\!\!\!\mid}^{r}_{Mc} b, \quad \text{and } b \mathop{\smile\!\!\!\!\!\!\mid}^{r}_{Ma''} c.$$

Abstract independence without base monotonicity

The previous theorem can be proven replacing $\mathop{\smile\!\!\!\!\!\!|}^r$ with any relation $\mathop{\smile\!\!\!\!\!\!|}^*$ satisfying:

1. Invariance: If $A \mathop{\smile\!\!\!\!\!\!|}^*_C B$ and $ABC \equiv A'B'C'$, then $A' \mathop{\smile\!\!\!\!\!\!|}^*_{C'} B'$.
2. Monotonicity: If $A \mathop{\smile\!\!\!\!\!\!|}^*_C B$, $A' \subseteq A$, and $B' \subseteq B$, then $A' \mathop{\smile\!\!\!\!\!\!|}^*_C B'$.
3. Symmetry: If $A \mathop{\smile\!\!\!\!\!\!|}^*_C B$, then $B \mathop{\smile\!\!\!\!\!\!|}^*_C A$.
4. Transitivity: Suppose $C \subseteq B \subseteq A$. If $A \mathop{\smile\!\!\!\!\!\!|}^*_B D$ and $B \mathop{\smile\!\!\!\!\!\!|}^*_C D$, then $A \mathop{\smile\!\!\!\!\!\!|}^*_C D$.
5. Normality: If $A \mathop{\smile\!\!\!\!\!\!|}^*_C B$, then $AC \mathop{\smile\!\!\!\!\!\!|}^*_C B$.
6. Full existence: For any A, B, and C, there exists $A' \equiv_C A$ such that $A' \mathop{\smile\!\!\!\!\!\!|}^*_C B$.
7. Finite character: If $A' \mathop{\smile\!\!\!\!\!\!|}^*_C B$ for all finite $A' \subseteq A$, then $A \mathop{\smile\!\!\!\!\!\!|}^*_C B$.
8. Strong local character: For every cardinal λ, there exists a cardinal κ such that for all A with $|A| = \lambda$, all B, and all $D \subseteq B$, there exists $D \subseteq C \subseteq B$ with $|C| \leq \max(|D|, \kappa)$ and $A \mathop{\smile\!\!\!\!\!\!|}^*_C B$.

Preservation of simplicity

Theorem (Kaplan–Ramsey)

T is simple if and only if T is NSOP$_1$ and $\mathop{\smile\!\!\!\!\!\!|}^K$ satisfies base monotonicity over models: for all $M \prec N \prec \mathbb{M}$, if $a \mathop{\smile\!\!\!\!\!\!|}^K_M Nb$, then $a \mathop{\smile\!\!\!\!\!\!|}^K_N b$.

Corollary

Suppose T_0 is stable with weak e.i. and 3-uniqueness. If T_i is simple and $\mathrm{ccl} = \mathrm{acl}_i$ for all $i \in I$, then $T^*_\cup$ is simple.

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Suppose T_0 is stable with weak e.i. and 3-uniqueness. If T_i is simple and $\mathrm{ccl} = \mathrm{acl}_i$ for all $i \in I$, then $T^*_\cup$ is simple.

Proof.

Fix $M \prec N \prec \mathbb{M}_\cup$ and $a \mathop{\smile\!\!\!\!\!\!\mid}^K_M Nb$. Then $\mathrm{ccl}(Ma) \mathop{\smile\!\!\!\!\!\!\mid}^K_M \mathrm{ccl}(Nb)$ in $\mathbb{M}_i$ for all i. Since T_i is simple, $a \mathop{\smile\!\!\!\!\!\!\mid}^f_M Nb$ in $\mathbb{M}_i$. Using base monotonicity for $\mathop{\smile\!\!\!\!\!\!\mid}^f$, $a \mathop{\smile\!\!\!\!\!\!\mid}^f_N b$, so $\mathrm{acl}_i(Na) \mathop{\smile\!\!\!\!\!\!\mid}^f_N \mathrm{acl}_i(Nb)$. Since $\mathrm{ccl} = \mathrm{acl}_i$, $\mathrm{ccl}(Na) \mathop{\smile\!\!\!\!\!\!\mid}^K_N \mathrm{ccl}(Nb)$ in $\mathbb{M}_i$. Thus $a \mathop{\smile\!\!\!\!\!\!\mid}^K_N b$ in $\mathbb{M}_\cup$, as desired. $\square$

Again, slightly weaker (but more technical) hypotheses suffice.

Questions / Future work

1. When does interpolative fusion preserve NTP_2? Elimination of imaginaries? NSOP? Rosiness? Non-maximality in the Keiseler order? Your favorite property here.
2. Interpolative fusions provide a rich source of examples around NSOP_1, which can be used to test conjectures and build intuition.
3. Complete the analogy:

   Simple is to NTP_2 as NSOP_1 is to X

   Property X should be preserved under interpolative fusions (over tame bases). So we already know examples of theories with Property X, e.g. Minh's theory of ACF_p with cyclically ordered multiplicative group.