Generic theories, independence, and $NSOP_1$

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Outline

- Background:
  - Forking independence and simple theories
  - Kim independence and $NSOP_1$ theories
- New $NSOP_1$ examples:
  - Generic $\mathcal{L}$-structures
  - Generic projective planes
- Preservation results:
  - Generic expansions
  - Generic Skolemizations

This is recent joint work:

- Alex Kruckman and Nicholas Ramsey, *Generic expansion and Skolemization in $NSOP_1$ theories*, arXiv:1706.06616, June 2017.
- Gabriel Conant and Alex Kruckman, *Independence in generic incidence structures*, arXiv:1709.09626, September 2017.

Notions of independence

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Fix a complete first-order theory T and a large highly saturated and homogeneous "monster model" $\mathbb{M} \models T$. By convention:

- All small models are elementary substructures $M \prec \mathbb{M}$.
- All small tuples b are tuples from $\mathbb{M}$.
- All small sets are subsets of $\mathbb{M}$.

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A "notion of independence" is presented as a ternary relation $\mathop{\smile\!\!\!\!\!\!\mid}$ on subsets of $\mathbb{M}$. For any small sets A, B, C, we read

$$A \mathop{\smile\!\!\!\!\!\!\mid}_C B \text{ as "} A \text{ is independent from } B \text{ over } C\text{".}$$

Algebraic independence ($\mathop{\smile\!\!\!\!\!\!\,|}^a$)

The most basic notion of independence is algebraic independence.

Definition

A formula $\varphi(x; a)$ is *algebraic* if it has only finitely many solutions. The *algebraic closure* of A, $\mathrm{acl}(A)$, is the set of all elements $b \in \mathbb{M}$ which satisfy some algebraic formula with parameters from A.

Define $A \mathop{\smile\!\!\!\!\!\!\,|}^a_C B \iff \mathrm{acl}(AC) \cap \mathrm{acl}(BC) = \mathrm{acl}(C)$.

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$$\text{Define } A \mathop{\smile\!\!\!\!\!\!|}^a_C B \iff \mathrm{acl}(AC) \cap \mathrm{acl}(BC) = \mathrm{acl}(C).$$

In any theory, $\mathop{\smile\!\!\!\!\!\!|}^a$ always satisfies some basic properties:

- Invariance: If $A \mathop{\smile\!\!\!\!\!\!|}^a_C B$ and $A'B'C' \equiv ABC$, then $A' \mathop{\smile\!\!\!\!\!\!|}^a_{C'} B'$.
- Symmetry: If $A \mathop{\smile\!\!\!\!\!\!|}^a_C B$, then $B \mathop{\smile\!\!\!\!\!\!|}^a_C A$.
- Monotonicity: If $A' \subseteq A$, $B' \subseteq B$, and $A \mathop{\smile\!\!\!\!\!\!|}^a_C B$, then $A' \mathop{\smile\!\!\!\!\!\!|}^a_C B'$.
- Existence: $A \mathop{\smile\!\!\!\!\!\!|}^a_C C$.
- Extension: If $A \mathop{\smile\!\!\!\!\!\!|}^a_C B$, and $B \subseteq B'$, then there exists $A' \equiv_{BC} A$ such that $A' \mathop{\smile\!\!\!\!\!\!|}^a_C B'$.

Dividing independence ($\mathop{\smile\!\!\!\!\!\!|}^d$) and forking independence ($\mathop{\smile\!\!\!\!\!\!|}^f$)

Definition (Shelah)

A formula $\varphi(x;b)$ *divides* over C if there is a C-indiscernible sequence $(b_i)_{i\in\omega}$ with $b_0 = b$ such that $\{\varphi(x;b_i) \mid i \in \omega\}$ is inconsistent.

Define $A \mathop{\smile\!\!\!\!\!\!|}^d_C B \iff$ no formula in $\mathrm{tp}(A/BC)$ divides over C.

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$\mathop{\smile\!\!\!\!\!\!\,|}^d$ may not satisfy extension. This motivates the following definition:

Definition (Shelah)

A formula $\varphi(x;b)$ *forks* over C if it implies a disjunction $\bigvee_{i=1}^n \psi_i(x;b_i)$ such that each formula $\psi_i(x;b_i)$ divides over C.

Define $A \mathop{\smile\!\!\!\!\!\!\,|}^f_C B \iff$ no formula in $\mathrm{tp}(A/BC)$ forks over C.

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Equivalently, $\mathop{\smile\!\!\!\!\!\!|}^f$ can be defined by "forcing" extension on $\mathop{\smile\!\!\!\!\!\!|}^d$:
$A \mathop{\smile\!\!\!\!\!\!|}^f_C B \iff$ for all $B' \supseteq B$, there exists $A' \equiv_{BC} A$ such that $A' \mathop{\smile\!\!\!\!\!\!|}^d_C B$.

Simple theories

Forking independence was originally defined in order to study stable theories. But Kim and Pillay showed that forking independence remains well-behaved and very useful in the wider class of simple theories.

Definition (Shelah '80)

A formula $\varphi(x;y)$ has the *tree property* if there exist tuples $(a_\eta)_{\eta\in\omega^{<\omega}}$ and $k \geq 2$ such that for all $\sigma \in \omega^\omega$, $\{\varphi(x; a_{\sigma|_n}) \mid n \in \omega\}$ is consistent, but for any $\eta \in \omega^{<\omega}$, $\{\varphi(x; a_{\eta\hat{\ }n}) \mid n \in \omega\}$ is k-inconsistent (meaning that any subset of size k is inconsistent).
T is *simple* if no formula has the tree property.

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Theorem (Kim '96)

- *T is simple if and only if $\mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$|$}}^f$ is symmetric: $A \mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$|$}}^f_C B \iff B \mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$|$}}^f_C A$.*
- *If T is simple, then $\mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$|$}}^f = \mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$|$}}^d$.*

Characterizing $\mathop{\smile\!\!\!\!\!\!|}^f$

Theorem (Kim–Pillay '96)

Let T be a complete theory and $\mathop{\smile\!\!\!\!\!\!|}$ any ternary relation on subsets of $\mathbb{M}$. Then T is simple and $\mathop{\smile\!\!\!\!\!\!|} = \mathop{\smile\!\!\!\!\!\!|}^f$ if and only if $\mathop{\smile\!\!\!\!\!\!|}$ satisfies:

- *Invariance.*
- *Symmetry.*
- *Existence.*
- *Extension.*
- *Full right-transitivity: If $D \subseteq C \subseteq B$, then $a \mathop{\smile\!\!\!\!\!\!|}_D C$ and $a \mathop{\smile\!\!\!\!\!\!|}_C B$ if and only if $a \mathop{\smile\!\!\!\!\!\!|}_D B$.*
- *Finite character: $a \mathop{\smile\!\!\!\!\!\!|}_C B$ if and only if for every finite tuple b from B, $a \mathop{\smile\!\!\!\!\!\!|}_C b$.*
- *Local character: For all a and B, there is $C \subseteq B$ such that $|C| \leq |T|$ and $a \mathop{\smile\!\!\!\!\!\!|}_C B$.*
- *... and* **the independence theorem***: see next slide.*

The independence theorem

The most important condition in the axiomatic characterization of forking independence in simple theories is the *independence theorem*:

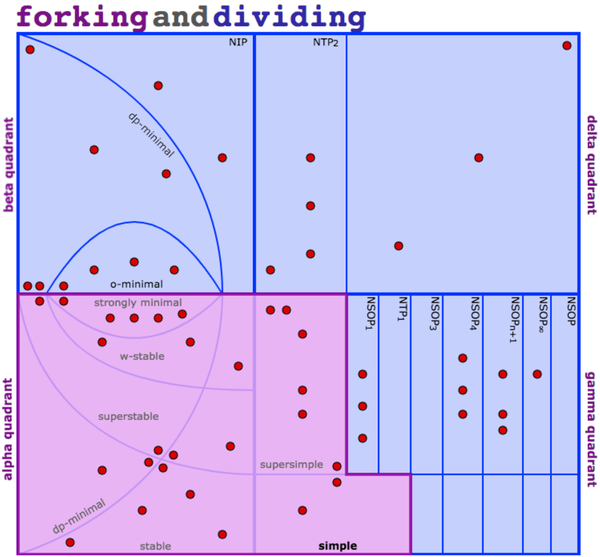
Let $M \prec \mathbb{M}$ be a model, A and B sets, and a and a' tuples. Suppose:

1. $a \equiv_M a'$,
2. $A \mathop{\smile\!\!\!\!\!\!\mid}_M B$,
3. $a \mathop{\smile\!\!\!\!\!\!\mid}_M A$, and
4. $a' \mathop{\smile\!\!\!\!\!\!\mid}_M B$.

Then there exists a'' such that:

1. $a'' \equiv_{MA} a$,
2. $a'' \equiv_{MB} a'$, and
3. $a'' \mathop{\smile\!\!\!\!\!\!\mid}_M AB$.

A map of the (first-order) universe



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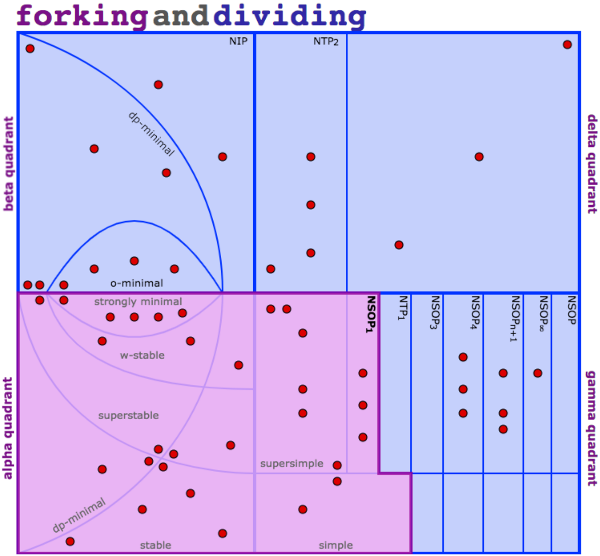

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Kim dividing

Definition

A global type $p(y) \in S_y(\mathbb{M})$ is *M-invariant* if for all formulas $\psi(y; z)$ and all $c \equiv_M c'$, $\psi(y; c) \in p \iff \psi(y; c') \in p$.

Fact: If $M \prec \mathbb{M}$, then every type $q(y) \in S_y(M)$ extends to a global M-invariant type (e.g. any coheir extension).

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Definition

If $p(y) \in S_y(\mathbb{M})$ is M-invariant, a *Morley sequence* over M for $p(y)$ is a sequence $(b_i)_{i\in\omega}$ such that $b_i \models p(y)|_{Mb_0\ldots b_{i-1}}$ for all i.

Fact: Such a Morley sequence $(b_i)_{i\in\omega}$ is M-indiscernible.

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Fact: Such a Morley sequence $(b_i)_{i\in\omega}$ is M-indiscernible.

Definition (Ramsey, after a suggestion of Kim)

A formula $\varphi(x, b)$ *Kim divides* over M if there is a global M-invariant type $p(y)$ extending $\text{tp}(b/M)$ and a Morley sequence $(b_i)_{i\in\omega}$ over M for $p(y)$ such that $\{\varphi(x, b_i) \mid i \in \omega\}$ is inconsistent.

NSOP_1 theories and Kim independence ($\mathop{\smile\!\!\!\!\!\!\;\;|}^K$)

Definition (Shelah '04)

A formula $\varphi(x;y)$ has the *stronger order property* 1 if there exist tuples $(a_\eta)_{\eta\in 2^{<\omega}}$ such that for all $\sigma \in 2^\omega$, $\{\varphi(x;a_{\sigma|_n}) \mid n \in \omega\}$ is consistent, but for any $\nu,\eta \in 2^{<\omega}$, if $\nu^\frown 0 \leq \eta$, then $\{\varphi(x;a_\eta), \varphi(x;a_{\nu^\frown 1})\}$ is inconsistent.
T is NSOP_1 if no formula has the stronger order property 1.

Snappy name forthcoming — for now, "NSOP_1".

NSOP_1 theories and Kim independence ($\mathop{\smile\!\!\!\!\!\!|}^K$)

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Snappy name forthcoming — for now, "NSOP$_1$".

Define $a \mathop{\smile\!\!\!\!\!\!|}^K_M b \iff$ no formula in tp(a/Mb) Kim divides over M.

Theorem (Kaplan–Ramsey '17)

- *T is NSOP$_1$ if and only if $\mathop{\smile\!\!\!\!\!\!|}^K$ is symmetric.*
- *If T is NSOP$_1$, $\mathop{\smile\!\!\!\!\!\!|}^K$ already satisfies extension, so "Kim forking" equals Kim dividing.*
- *In any theory, $\mathop{\smile\!\!\!\!\!\!|}^f_M \Rightarrow \mathop{\smile\!\!\!\!\!\!|}^K_M$. In simple theories, $\mathop{\smile\!\!\!\!\!\!|}^f_M \Leftrightarrow \mathop{\smile\!\!\!\!\!\!|}^K_M$.*

Characterizing $\mathop{\smile\!\!\!\!\!\!\mid}^K$

Crucially, there is a Kim–Pillay style characterization of $\mathop{\smile\!\!\!\!\!\!\mid}^K$ in NSOP$_1$.

Theorem (Kaplan–Ramsey '17)

Let T be a complete theory and $\mathop{\smile\!\!\!\!\!\!\mid}$ any ternary relation on subsets of $\mathbb{M}$. Then T is NSOP$_1$ and $\mathop{\smile\!\!\!\!\!\!\mid}_M = \mathop{\smile\!\!\!\!\!\!\mid}^K_M$ for all $M \prec \mathbb{M}$ if and only if $\mathop{\smile\!\!\!\!\!\!\mid}_M$ satisfies:

1. *Invariance: If $A \mathop{\smile\!\!\!\!\!\!\mid}_M B$ and $A'B'M' \equiv ABM$, then $A' \mathop{\smile\!\!\!\!\!\!\mid}_{M'} B'$.*
2. *Symmetry: If $A \mathop{\smile\!\!\!\!\!\!\mid}_M B$, then $B \mathop{\smile\!\!\!\!\!\!\mid}_M A$.*
3. *Monotonicity: If $A' \subseteq A$, $B' \subseteq B$, and $A \mathop{\smile\!\!\!\!\!\!\mid}_M B$, then $A' \mathop{\smile\!\!\!\!\!\!\mid}_M B'$.*
4. *Existence: $A \mathop{\smile\!\!\!\!\!\!\mid}_M M$.*
5. *Strong finite character and witnessing: if $A \not\mathop{\smile\!\!\!\!\!\!\mid}_M B$, then there is a formula $\varphi(x;b) \in \mathrm{tp}(A/MB)$ such that for any $a' \models \varphi(x;b)$, $a' \not\mathop{\smile\!\!\!\!\!\!\mid}_M b$. Moreover, $\varphi(x;b)$ Kim divides over M.*
6. **The independence theorem.**

Deficiencies of $\mathop{\smile\!\!\!\!\!\!|}^K$ in unsimple theories

A key property of $\mathop{\smile\!\!\!\!\!\!|}^f$ which is lost by $\mathop{\smile\!\!\!\!\!\!|}^K$ in properly NSOP$_1$ theories is:

- Base monotonicity: If $D \subseteq C \subseteq B$ and $A \mathop{\smile\!\!\!\!\!\!|}^f_D B$, then $A \mathop{\smile\!\!\!\!\!\!|}^f_C B$.

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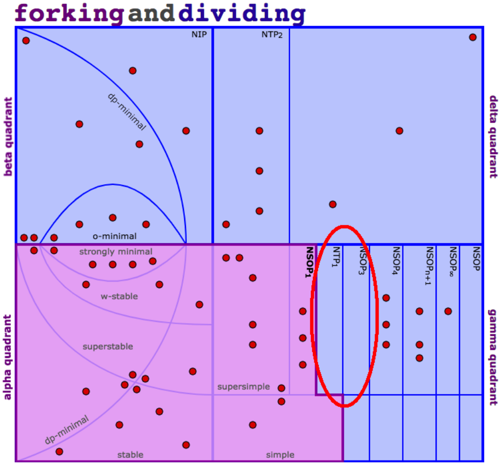
- Base monotonicity: If $D \subseteq C \subseteq B$ and $A \mathop{\smile\!\!\!\!\!\!\mid}^f_D B$, then $A \mathop{\smile\!\!\!\!\!\!\mid}^f_C B$.

Also, we currently only know how to define $A \mathop{\smile\!\!\!\!\!\!\mid}^K_M B$ when $M \prec \mathbb{M}$.

Why? Using the definition verbatim, $A \mathop{\smile\!\!\!\!\!\!\mid}^K_C B$ is vacuously true if $\mathrm{tp}(B/C)$ does not extend to a global C-invariant type.

Returning to the map



Using the new criterion for NSOP_1, Chernikov, Ramsey, and others have shown that all known examples of NSOP_3 theories are NSOP_1.

Generic $\mathcal{L}$-structures

Fact: In any language $\mathcal{L}$, the empty $\mathcal{L}$-theory has a model companion $T^{\emptyset}_{\mathcal{L}}$, the theory of existentially closed $\mathcal{L}$-structures.

We call $T^{\emptyset}_{\mathcal{L}}$ the *generic theory of $\mathcal{L}$-structures*.

It is well-known that if $\mathcal{L}$ is relational, then $T^{\emptyset}_{\mathcal{L}}$ is simple.

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It is well-known that if $\mathcal{L}$ is relational, then $T^{\emptyset}_{\mathcal{L}}$ is simple.

In an unpublished preprint, Jeřábek showed that $T^{\emptyset}_{\mathcal{L}}$ is NSOP_3 for any language $\mathcal{L}$, and he asked:

Question

Is $T^{\emptyset}_{\mathcal{L}}$ NSOP_1? Does it have weak elimination of imaginaries?

(Later, Jeřábek independently answered these questions.)

Generic binary functions

If $\mathcal{L}$ contains a single binary function f, then already $T^{\emptyset}_{\mathcal{L}}$ is not simple.

Definition

A formula $\varphi(x;y)$ has the *tree property* 2 (TP$_2$) if there exist tuples $(a_{i,j})_{i,j<\omega}$ such that for all $\sigma \in \omega^\omega$, $\{\varphi(x;a_{n,\sigma(n)}) \mid n < \omega\}$ is consistent, but for any $n < \omega$ and $i < j < \omega$, $\{\varphi(x;a_{n,i}), \varphi(x;a_{n,j})\}$ is inconsistent. T is NTP$_2$ if no formula has TP$_2$.

The formula $\varphi(x;y_1,y_2)$: $f(x,y_1) = y_2$ has TP$_2$.

Let $(b_i)_{i<\omega}$ and $(c_{i,j})_{i,j<\omega}$ be distinct, and set $a_{i,j} = (b_i, c_{i,j})$.

- $\{f(x,b_n) = c_{n,\sigma(n)}\}$ is consistent, while
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If $b \mathop{\smile\!\!\!\!\!\!|}^{a}_{M} c$, then $\varphi(x; b, c)$ divides over M (along an M-indiscernible sequence $(b, c_i)_{i\in\omega}$ with $c_i \neq c_j$ for all $i < j$) but does not Kim-divide over M (since if $(b_i, c_i)_{i\in\omega}$ is a Morley sequence, the b_i are all distinct).

Classifying $T^\emptyset_\mathcal{L}$

Theorem (K.–Ramsey, independently Jeřábek)

For any language $\mathcal{L}$:

- $T^\emptyset_\mathcal{L}$ *eliminates quantifiers, and* $\mathrm{acl}(A) = \langle A \rangle$*, the substructure generated by* A*.*
- $\mathop{\smile\!\!\!\!\!\vert}^a$ *satisfies the independence theorem over arbitrary sets.*
- *It follows easily that* $T^\emptyset_\mathcal{L}$ *is NSOP*$_1$ *and* $\mathop{\smile\!\!\!\!\!\vert}^K = \mathop{\smile\!\!\!\!\!\vert}^a$ *over models.*

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- *It follows easily that* $T^\emptyset_\mathcal{L}$ *is NSOP*$_1$ *and* $\mathop{\smile\!\!\!\!\!\!|}^K = \mathop{\smile\!\!\!\!\!\!|}^a$ *over models.*

Jeřábek's preprint contains a complete classification of $T^\emptyset_\mathcal{L}$:

Relation arities:	≤ 0	≤ 1	any	any
Function arities:	≤ 0	≤ 1	≤ 1	any
$T^\emptyset_\mathcal{L}$ is:	strongly minimal	stable*	simple*	NSOP$_1$

* If $T^\emptyset_\mathcal{L}$ is stable/simple, then it is superstable/supersimple if and only if there is at most one unary function symbol in L.

Projective planes

An *incidence structure* is a structure in the language $\{P, L, I\}$, where:

- P and L are unary relation partitioning the structure into two disjoint sets ("points" and "lines")
- I is a binary relation ("incidence") such that if $I(a, b)$ holds, then $a \in P$ and $b \in L$.

In other words, an incidence structure is a bipartite graph with the two halves of the partition named.

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An incidence structure A is a *partial plane* if any two points are incident with *at most* one line and any two lines are incident with *at most* one point. Let $T^p_{2,2}$ be the theory of *partial* planes.

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An incidence structure A is a **projective plane** if any two points are incident with **exactly** one line and any two lines are incident with **exactly** one point. Let $T^c_{2,2}$ be the theory of **projective** planes.

Incidence structures and bipartite graphs

Equivalently, an incidence structure is a partial plane if it does not contain a copy of the complete bipartite graph $K_{2,2}$.

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Equivalently, an incidence structure is a partial plane if it does not contain a copy of the complete bipartite graph $K_{2,2}$.

Almost everything I will say can be generalized to $T^p_{m,n}$ the theory of incidence structures which do not contain a copy of $K_{m,n}$, for $m, n \geq 2$.

Note that the model companion $T_{m,n}$ of $T^p_{m,n}$ can be viewed as bipartite analogues of the Henson theories T_n (T_n is the generic theory of graphs which do not contain a copy of the complete graph K_n).

In this talk, I'll stick to projective planes for simplicity.

Generic projective planes

For any subset A of a projective plane B, there is a smallest projective plane containing it, called its I-*closure*, obtained by iteratively adding the intersection points of all pairs of lines, and the connecting lines of all pairs of points.

Definition

A formula $\varphi(\overline{x})$ is *basic existential* if it has the form $\exists \overline{y} \bigwedge_{\psi \in p} \psi(\overline{x}, \overline{y})$, where $p(\overline{x}, \overline{y})$ is a complete quantifier-free type which implies that $\overline{y}$ is in the I-closure of $\overline{x}$.

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Theorem (Conant–K.)

$T^p_{2,2}$ has a model companion $T_{2,2}$, which is also the model companion of $T^c_{2,2}$. In $T_{2,2}$, acl *coincides with I-closure, and $T_{2,2}$ eliminates quantifiers "down to the I-closure": every formula is equivalent to a disjunction of basic existential formulas.*

Independence and NSOP$_1$

Define $A \mathop{\smile\!\!\!\!\!\!|}^{I}_{C} B$ if and only if $A \mathop{\smile\!\!\!\!\!\!|}^{a}_{C} B$ and there are no incidences between $\mathrm{acl}(AC) \setminus \mathrm{acl}(C)$ and $\mathrm{acl}(BC) \setminus \mathrm{acl}(C)$.

Just as in the case of $T^{\emptyset}_{\mathcal{L}}$:

Theorem (Conant–K.)

- $\mathop{\smile\!\!\!\!\!\!|}^{I}$ *satisfies the independence theorem over arbitrary sets.*
- *It follows easily that* $T_{2,2}$ *is NSOP*$_1$ *and* $\mathop{\smile\!\!\!\!\!\!|}^{K} = \mathop{\smile\!\!\!\!\!\!|}^{I}$ *over models.*

Independence and NSOP$_1$

Define $A \mathop{\smile\!\!\!\!\!\!|}^{I}_{C} B$ if and only if $A \mathop{\smile\!\!\!\!\!\!|}^{a}_{C} B$ and there are no incidences between $\mathrm{acl}(AC) \setminus \mathrm{acl}(C)$ and $\mathrm{acl}(BC) \setminus \mathrm{acl}(C)$.

Just as in the case of $T^{\emptyset}_{\mathcal{L}}$:

Theorem (Conant–K.)

- $\mathop{\smile\!\!\!\!\!\!|}^{I}$ *satisfies the independence theorem over arbitrary sets.*
- *It follows easily that $T_{2,2}$ is NSOP$_1$ and $\mathop{\smile\!\!\!\!\!\!|}^{K} = \mathop{\smile\!\!\!\!\!\!|}^{I}$ over models.*

So $T_{2,2}$ is tamer (in a sense) than the Henson theories T_n, which have SOP$_3$ when $n \geq 3$.

On the other hand, T_n is $\aleph_0$-categorical, but $T_{2,2}$ is not – acl is not locally finite.

Failure of simplicity

Let p_1 and p_2 be points and ℓ_1 and ℓ_2 lines such that there are no incidences between the p_i and ℓ_j, but the unique line ℓ^* through p_1 and p_2 contains the unique point p^* at the intersection of ℓ_1 and ℓ_2.

Letting $\varphi(x_1, x_2, x^*, y_1, y_2, y^*)$ be the conjunction of all atomic formulas satisfied by $(p_1, p_2, p^*, \ell_1, \ell_2, \ell^*)$, one can show that the formula $\psi(x_1, y_1; x_2, y_2)\colon \exists x^* \, \exists y^* \, \varphi(x_1, x_2, x^*, y_1, y_2, y^*)$ has TP_2.

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Also, $p_1\ell_1 \mathop{\smile\!\!\!\!\!\!|}^I_{\emptyset} p_2\ell_2$. But $p^* \in \mathrm{acl}(p_1\ell_1\ell_2)$, so $\ell^* \in \mathrm{acl}(p_1\ell_1\ell_2)$, and ℓ^* is incident to p_2. Thus $p_1\ell_1 \not\mathop{\smile\!\!\!\!\!\!|}^I_{\ell_2} p_2\ell_2$, and $\mathop{\smile\!\!\!\!\!\!|}^I$ fails base monotonicity.

Characterizing dividing

Now I'll make some remarks about $T_{\mathcal{L}}^{\emptyset}$ and $T_{2,2}$ in parallel.

Theorem (K.–Ramsey)

- *In* $T_{\mathcal{L}}^{\emptyset}$*,* $\mathop{\smile\!\!\!\!\!\!|}^{d}$ *is obtained by "forcing" base monotonicity on* $\mathop{\smile\!\!\!\!\!\!|}^{a}$*:* $A \mathop{\smile\!\!\!\!\!\!|}^{d}_{C} B$ *if and only if* $A \mathop{\smile\!\!\!\!\!\!|}^{a}_{C'} B$ *for all* $C \subseteq C' \subseteq \mathrm{acl}(BC)$*.*
- *There is a formula which forks but does not divide over* $\emptyset$*. For example, if* f *is a binary function symbol, take the formula* $f(x,b) = c \vee x = b$*, where* $b \notin \mathrm{acl}(\emptyset)$ *and* $c \notin \mathrm{acl}(b)$*.*

Characterizing dividing

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- *There is a formula which forks but does not divide over $\emptyset$. For example, if f is a binary function symbol, take the formula* $f(x,b) = c \vee x = b$, *where* $b \notin \mathrm{acl}(\emptyset)$ *and* $c \notin \mathrm{acl}(b)$.

Theorem (Conant–K.)

- *In $T_{2,2}$, $\mathop{\smile\!\!\!\!\!\!|}^d$ is obtained by "forcing" base monotonicity on $\mathop{\smile\!\!\!\!\!\!|}^I$:* $A \mathop{\smile\!\!\!\!\!\!|}^d_C B$ *if and only if* $A \mathop{\smile\!\!\!\!\!\!|}^I_{C'} B$ *for all* $C \subseteq C' \subseteq \mathrm{acl}(BC)$.
- *If $\varphi(x_1, y_1; x_2, y_2)$ be the formula from the last slide, then the formula* $\varphi(x_1, y_1; p_2, \ell_2) \vee x_1 = p_2 \vee y_1 = \ell_2$ *forks but does not divide over $\emptyset$.*

Forking and dividing

However, in both $T_{\mathcal{L}}^{\emptyset}$ and $T_{2,2}$, $\mathop{\smile\!\!\!\!\!\!|}^{f} = \mathop{\smile\!\!\!\!\!\!|}^{d}$, i.e. forking = dividing for complete types. In both cases, we follow a common strategy:

Forking and dividing

However, in both $T^{\emptyset}_{\mathcal{L}}$ and $T_{2,2}$, $\mathop{\smile\!\!\!\!\!\!|}^{f} = \mathop{\smile\!\!\!\!\!\!|}^{d}$, i.e. forking = dividing for complete types. In both cases, we follow a common strategy:

Lemma

Suppose $\mathop{\smile\!\!\!\!\!\!|}^{\otimes}$ is a ternary relation satisfying existence and extension, and such that for all sets A and $B \subseteq C \subseteq D$, we have

$$A \mathop{\smile\!\!\!\!\!\!|}^{d}_{D} C \text{ and } A \mathop{\smile\!\!\!\!\!\!|}^{\otimes}_{C} B \implies A \mathop{\smile\!\!\!\!\!\!|}^{d}_{D} B.$$

Then $\mathop{\smile\!\!\!\!\!\!|}^{f} = \mathop{\smile\!\!\!\!\!\!|}^{d}$.

Proof.

It suffices to show that $\mathop{\smile\!\!\!\!\!\!|}^{d}$ satisfies extension. Given $A \mathop{\smile\!\!\!\!\!\!|}^{d}_{D} C$ and $C \subseteq B$, we have $A \mathop{\smile\!\!\!\!\!\!|}^{\otimes}_{C} C$ (by existence), so there is some $A' \equiv_C A$ such that $A' \mathop{\smile\!\!\!\!\!\!|}^{\otimes}_{C} B$ (by extension). By invariance, also $A' \mathop{\smile\!\!\!\!\!\!|}^{d}_{D} C$. So $A' \mathop{\smile\!\!\!\!\!\!|}^{d}_{D} B$. □

Defining $\mathop{\smile\!\!\!\!\!\!\,|}^{\otimes}$

In $T_{\mathcal{L}}^{\emptyset}$, define $A \mathop{\smile\!\!\!\!\!\!\,|}^{\otimes}_C B$ if and only if $\mathrm{acl}(ABC) \cong AC \otimes_C BC$, the fibered coproduct of AC and BC over C in the category of $\mathcal{L}$-structures (i.e. the $\mathcal{L}$-structure freely generated by AC and BC over C).

Defining $\mathop{\smile\!\!\!\!\!\!|}^{\otimes}$

In $T^{\emptyset}_{\mathcal{L}}$, define $A \mathop{\smile\!\!\!\!\!\!|}^{\otimes}_C B$ if and only if $\mathrm{acl}(ABC) \cong AC \otimes_C BC$, the fibered coproduct of AC and BC over C in the category of $\mathcal{L}$-structures (i.e. the $\mathcal{L}$-structure freely generated by AC and BC over C).

In $T_{2,2}$, there is also a natural notion of "free generation" (due to Marshall Hall in 1943). Given any partial plane $A \models A^{p}_{2,2}$:

- Define $A_0 = A$.
- A pair of lines in A_n is *open* if they do not have an intersection point in A_n, and a pair of points in A_n is *open* if they do not have a connecting line.
- Let A_{n+1} be the partial plane obtained by adding an intersection point to each open pair of lines in A_n and a connecting line to each open pair of lines in A_n.
- Set $F(A) = \bigcup_{n\in\omega} A_n$, the *free completion* of A.

Define $A \mathop{\smile\!\!\!\!\!\!|}^{\otimes}_C B$ if and only if $\mathrm{acl}(ABC) \cong F(\mathrm{acl}(AB) \cup \mathrm{acl}(AC))$.

Weak elimination of imaginaries

Recall that T has *weak elimination of imaginaries* if for every imaginary $e \in \mathbb{M}^{\mathrm{eq}} \models T^{\mathrm{eq}}$, there exists a real tuple $a \in \mathbb{M}$ such that $e \in \mathrm{dcl}^{\mathrm{eq}}(a)$ and $a \in \mathrm{acl}^{\mathrm{eq}}(e)$.

Lemma (Montenegro–Rideau)

Suppose there is a ternary relation $\mathop{\smile\!\!\!\!\!|}$ *on* $\mathbb{M} \models T$*, satisfying the following properties:*

(i) *Given* $a, b \in \mathbb{M}$ *and* $C^* = \mathrm{acl}^{\mathrm{eq}}(C^*) \subset \mathbb{M}^{\mathrm{eq}}$*, and letting* $C = C^* \cap \mathbb{M}$*, there exists* $a' \equiv_{C^*} a$ *such that* $a' \mathop{\smile\!\!\!\!\!|}_C b$*.*

(ii) *Given* $a, b, c \in \mathbb{M}$ *and* $C = \mathrm{acl}(C) \subset \mathbb{M}$ *such that* $a \equiv_C b$*,* $b \mathop{\smile\!\!\!\!\!|}_C a$*, and* $c \mathop{\smile\!\!\!\!\!|}_C a$*, there exists* c' *such that* $c'a \equiv_C c'b \equiv_C ca$*.*

Then T *has weak elimination of imaginaries.*

Weak elimination of imaginaries

(i) Given $a, b \in \mathbb{M}$ and $C^* = \mathrm{acl}^{\mathrm{eq}}(C^*) \subset \mathbb{M}^{\mathrm{eq}}$, and letting $C = C^* \cap \mathbb{M}$, there exists $a' \equiv_{C^*} a$ such that $a' \mathop{\smile\!\!\!\!\!\mid}_C b$.

(ii) Given $a, b, c \in \mathbb{M}$ and $C = \mathrm{acl}(C) \subset \mathbb{M}$ such that $a \equiv_C b$, $b \mathop{\smile\!\!\!\!\!\mid}_C a$, and $c \mathop{\smile\!\!\!\!\!\mid}_C a$, there exists c' such that $c'a \equiv_C c'b \equiv_C ca$.

Proof.

Fix $e \in \mathbb{M}^{\mathrm{eq}}$, and let $C^* = \mathrm{acl}^{\mathrm{eq}}(e)$ and $C = C^* \cap \mathbb{M}$. It suffices to show that $e \in \mathrm{dcl}^{\mathrm{eq}}(C)$, so pick some $\sigma \in \mathrm{Aut}(\mathbb{M}^{\mathrm{eq}}/C)$ and show $\sigma(e) = e$.

Pick $a \in \mathbb{M}$ and a $\emptyset$-definable function f such that $f(a) = e$. By (i), there exist $b \equiv_{C^*} \sigma(a)$ and $c \equiv_{C^*} a$ such that $b \mathop{\smile\!\!\!\!\!\mid}_C a$ and $c \mathop{\smile\!\!\!\!\!\mid}_C a$. Note that $f(c) = e$. Since $a \equiv_C \sigma(a) \equiv_C b$, we apply (ii) to find c' such that $c'a \equiv_C c'b \equiv_C ca$. Now $f(a) = f(c)$ implies $f(a) = f(c')$ and $f(b) = f(c')$. So $f(b) = e$, which implies $f(\sigma(a)) = e$, so $\sigma(e) = e$. □

Weak elimination of imaginaries

We seek to apply the Lemma to $\mathop{\smile\!\!\!\!\!\!\mid}^a$ and $\mathop{\smile\!\!\!\!\!\!\mid}^I$ in our theories.
(ii) follows from the independence theorem, so it suffices to show (i):

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When $\mathop{\smile\!\!\!\!\!\!|} = \mathop{\smile\!\!\!\!\!\!|}^a$, this is not hard: Use extension for $\mathop{\smile\!\!\!\!\!\!|}^a$ in $(T_{\mathcal{L}}^{\emptyset})^{\mathrm{eq}}$ to find $a' \equiv_{C^*} a$ such that $a' \mathop{\smile\!\!\!\!\!\!|}^a_{C^*} b$ in $\mathbb{M}^{\mathrm{eq}}$. Intersecting with $\mathbb{M}$, $a' \mathop{\smile\!\!\!\!\!\!|}^a_C b$.

In $T_{2,2}$, we don't have a version of $\mathop{\smile\!\!\!\!\!\!|}^I$ in $(T_{2,2})^{\mathrm{eq}}$. Instead, we show that if we take a large $\mathop{\smile\!\!\!\!\!\!|}^a$-independent (in $\mathbb{M}^{eq}$) array in $\mathrm{tp}(b/C^*)$, we must have $a \mathop{\smile\!\!\!\!\!\!|}^I_C b'$ for some b' in the array.

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Theorem (K.–Ramsey, Conant–K.)

$T^{\emptyset}_L$ and $T_{2,2}$ have weak elimination of imaginaries.

Weak elimination of imaginaries

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Theorem (K.–Ramsey, Conant–K.)

$T^{\emptyset}_L$ and $T_{2,2}$ have weak elimination of imaginaries.

Having eliminated imaginaries, we can also show that forking and thorn forking coincide in these theories, so neither is rosy.

Adding generic structure

Recipe:

1. Start with a base $\mathcal{L}$-theory T.
2. Add new symbols: $\mathcal{L} \subseteq \mathcal{L}_{\text{new}}$.
3. And new axioms governing them: $T \subseteq T_{\text{new}}$.
4. Take the model companion (if it exists): T^*_{new}.

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Example 0: Generic automorphisms

- $\mathcal{L}_{\text{new}} = \mathcal{L} \cup \{\sigma\}$, a unary function symbol.
- $T_{\text{new}} = T \cup$ "σ is an $\mathcal{L}$-automorphism".
- $T^*_{\text{new}} = T_A$, the theory T with a generic automorphism

[e.g. if $T = \text{ACF}$, then $T_A = \text{ACFA}$]

The question of whether T_A exists is often nontrivial.

Example 1: Generic expansions

- $\mathcal{L}_{\text{new}} = \mathcal{L}'$, any expansion of $\mathcal{L}$ by new constant, function, and relation symbols.
- $T_{\text{new}} = T$, so the new symbols are interpreted arbitrarily.
- $T^*_{\text{new}} = T_{\mathcal{L}'}$, the generic expansion of T to $\mathcal{L}'$.

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- $T^*_{\text{new}} = T_{\mathcal{L}'}$, the generic expansion of T to $\mathcal{L}'$.

Theorem (Winkler '75)

If T is model complete and eliminates $\exists^\infty$, then $T_{\mathcal{L}'}$ exists.

Generic Skolemizations

Definition

A definable function $f_\varphi(\overline{y})$ is a *Skolem function* for the formula $\varphi(x;\overline{y})$ if $\mathbb{M} \models \varphi(f_\varphi(\overline{a}),\overline{a})$ whenever $\varphi(\mathbb{M},\overline{a})$ is nonempty.

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Example 2: Generic Skolemizations

- $\mathcal{L}_{\text{new}} = \mathcal{L}_{\text{Sk}} = \mathcal{L} \cup \{f_\varphi \mid \varphi(x;\overline{y})$ an $\mathcal{L}$-formula$\}$.
- $T_{\text{new}} = T \cup \{\forall \overline{y}\,(\exists x\, \varphi(x;\overline{y}) \to \varphi(f_\varphi(\overline{y});\overline{y})) \mid \varphi(x;\overline{y})$ an $\mathcal{L}$-formula$\}$.
- $T^*_{\text{new}} = T_{\text{Sk}}$, the generic Skolemization of T.

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Preservation results

For the rest of this talk, assume T is model complete and eliminates $\exists^\infty$.

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Theorem (Chatzidakis–Pillay '98)

- *If T is stable and T_A exists, then T_A is simple.*
- *If T is simple and $\mathcal{L}' = \mathcal{L} \cup \{P\}$, where P is a unary relation symbol, then $T_{\mathcal{L}'}$ is simple.*

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Theorem (Nübling '03)

- *Let T^a_{Sk} be the theory obtained by adding generic Skolem functions for algebraic formulas only. If T is simple, then T^a_{Sk} is simple.*
- *If T is simple with QE, $\mathrm{acl}(A) = A$ for all sets A, and $\mathcal{L}' = \mathcal{L} \cup \{f\}$, where f is a unary function symbol, then $T_{\mathcal{L}'}$ is simple.*

Preservation results

Each of the results on the previous slide (except the last, which Nübling proves by counting types), has the following proof strategy:

- Let T' be the generic theory expanding T, let $\mathbb{M}' \models T'$ be a monster model, and let $\mathbb{M} \models T$ be its reduct to $\mathcal{L}$.
- Define a notion of independence in T' in terms of independence in T:

$$a \underset{C}{\downarrow} b \text{ in } \mathbb{M}' \iff \mathrm{acl}_{T'}(Ca) \underset{C}{\downarrow^{f}} \mathrm{acl}_{T'}(Cb) \text{ in } \mathbb{M}.$$

- Apply the Kim–Pillay theorem characterizing $\downarrow^{f}$ in simple theories. The main difficulty is checking the independence theorem.

New preservation results

Theorem (K.–Ramsey)

- *For any* $\mathcal{L}' \supseteq \mathcal{L}$, *if* T *is NSOP*$_1$, *then* $T_{\mathcal{L}'}$ *is NSOP*$_1$. *Further, for* $\mathbb{M}' \models T_{\mathcal{L}'}$ *and* $\mathbb{M} \models T$ *its reduct to* $\mathcal{L}$,

$$a \mathop{\smile\!\!\!\!\!\!\;|}^{K}_{M} b \text{ in } \mathbb{M}' \iff \mathrm{acl}_{T_{\mathcal{L}'}}(Ma) \mathop{\smile\!\!\!\!\!\!\;|}^{K}_{M} \mathrm{acl}_{T_{\mathcal{L}'}}(Mb) \text{ in } \mathbb{M}.$$

- *If* T *is NSOP*$_1$, *then* T_{Sk} *is NSOP*$_1$. *Further, for* $\mathbb{M}' \models T_{\mathrm{Sk}}$ *and* $\mathbb{M} \models T$ *its reduct to* $\mathcal{L}$,

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Proof strategy: Apply the Kaplan–Ramsey theorem characterizing $\mathop{\smile\!\!\!\!\!\!\;|}^{K}$ in NSOP$_1$ theories. Again, the main difficulty is the independence theorem. The proof involves some technical work on the relationship between $\mathop{\smile\!\!\!\!\!\!\;|}^{K}$ and $\mathop{\smile\!\!\!\!\!\!\;|}^{a}$ in arbitrary NSOP$_1$ theories.

$\mathop{\smile\!\!\!\!\!\!\mid}^K$ and $\mathop{\smile\!\!\!\!\!\!\mid}^a$

Theorem (Kaplan–Ramsey)

If T is NSOP$_1$, $\mathop{\smile\!\!\!\!\!\!\mid}^K$ satisfies the following properties:

1. *Extension: if $a \mathop{\smile\!\!\!\!\!\!\mid}^K_M b$, then for any c, there exists a' such that $a' \equiv_{Mb} a$ and $a' \mathop{\smile\!\!\!\!\!\!\mid}^K_M bc$.*
2. *The chain condition: if $a \mathop{\smile\!\!\!\!\!\!\mid}^K_M b$ and $I = (b_i)_{i<\omega}$ is a Morley sequence over M in a global M-invariant type extending $\mathrm{tp}(b/M)$, then there exists a' such that $a' \equiv_{Mb} a$, $a' \mathop{\smile\!\!\!\!\!\!\mid}^K_M I$, and I is Ma'-indiscernible.*
3. *The independence theorem: if $a \mathop{\smile\!\!\!\!\!\!\mid}^K_M b$, $a' \mathop{\smile\!\!\!\!\!\!\mid}^K_M c$, $b \mathop{\smile\!\!\!\!\!\!\mid}^K_M c$, and $a \equiv_M a'$, then there exists a'' such that $a'' \equiv_{Mb} a$, $a'' \equiv_{Mc} a$, and $a'' \mathop{\smile\!\!\!\!\!\!\mid}^K_M bc$.*

Theorem (K.–Ramsey)

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$\mathop{\smile}\limits^{K}$ and $\mathop{\smile}\limits^{a}$

Theorem (K.–Ramsey)

If T is NSOP$_1$, $\mathop{\smile}\limits^{K}$ satisfies the following properties:

1. *Extension: if $a \mathop{\smile}\limits^{K}_M b$, then for any c, there exists a' such that $a' \equiv_{Mb} a$ and $a' \mathop{\smile}\limits^{K}_M bc$,* **and further,** $a' \mathop{\smile}\limits^{a}_{Mb} c$**.**
2. *The chain condition: if $a \mathop{\smile}\limits^{K}_M b$ and $I = (b_i)_{i<\omega}$ is a Morley sequence over M in a global M-invariant type extending $\mathrm{tp}(b/M)$, then there exists a' such that $a' \equiv_{Mb} a$, $a' \mathop{\smile}\limits^{K}_M I$, and I is Ma'-indiscernible,* **and further,** $b_i \mathop{\smile}\limits^{a}_{Ma'} b_j$ **for all** $i \neq j$**.**
3. *The independence theorem: if $a \mathop{\smile}\limits^{K}_M b$, $a' \mathop{\smile}\limits^{K}_M c$, $b \mathop{\smile}\limits^{K}_M c$, and $a \equiv_M a'$, then there exists a'' such that $a'' \equiv_{Mb} a$, $a'' \equiv_{Mc} a$, and $a'' \mathop{\smile}\limits^{K}_M bc$,* **and further,** $a'' \mathop{\smile}\limits^{a}_{Mb} c$**,** $a'' \mathop{\smile}\limits^{a}_{Mc} b$**, and** $b \mathop{\smile}\limits^{a}_{Ma''} c$**.**

If T is simple (so $\mathop{\smile}\limits^{K} = \mathop{\smile}\limits^{f}$), all of the **and further** clauses are easy, e.g.:

$$a' \mathop{\smile}\limits^{f}_{M} bc \implies a' \mathop{\smile}\limits^{f}_{Mb} c \implies a' \mathop{\smile}\limits^{a}_{Mb} c$$

Built-in Skolem functions

The generic Skolemization T_{Sk} has a Skolem function for every $\mathcal{L}$-formula, but not necessarily for every $\mathcal{L}_{\mathrm{Sk}}$-formula. But we can iterate the construction to obtain an expansion with Skolem functions for all formulas.

Corollary (K.–Ramsey)

Any NSOP$_1$ theory T which eliminates $\exists^\infty$ has an expansion to an NSOP$_1$ theory T_{Sk}^∞ in a language $\mathcal{L}_{\mathrm{Sk}}^\infty$ with built-in Skolem functions. Further, for $\mathbb{M}_{\mathrm{Sk}}^\infty \models T_{\mathrm{Sk}}^\infty$ and $\mathbb{M} \models T$ its reduct to $\mathcal{L}$,

$$a \mathop{\smile\!\!\!\!\!|}_{M}^{K} b \text{ in } \mathbb{M}_{\mathrm{Sk}}^\infty \iff \mathrm{acl}_{T_{\mathrm{Sk}}^\infty}(Ma) \mathop{\smile\!\!\!\!\!|}_{M}^{K} \mathrm{acl}_{T_{\mathrm{Sk}}^\infty}(Mb) \text{ in } \mathbb{M}.$$

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This result may turn out to be a useful technical tool: in an NSOP$_1$ theory with built-in Skolem functions, $\mathop{\smile\!\!\!\!\!\!|}^{K}$ makes sense over an arbitrary base C, since $\mathrm{acl}(C)$ is a model.