# Model Theory Lecture Notes

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## Contents

# 1 Syntax and semantics

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Mathematical logic, broadly speaking, is the field of mathematics which is concerned with the language we use to describe and reason about mathematical objects. That is, it highlights the distinction between syntax (the language) and semantics (the object itself). This may sound philosophical, but mathematicians constantly describe objects of interest by syntactic presentations, and then reason about the object by manipulating the syntax, e.g. the description of an algebraic variety as the locus of some system of polynomials, the coordinatization of a manifold by explicit charts, the specification of a construction or function by an algorithm, the presentation of a group by generators and relations, or the description of a function as an infinite series.

There are many kinds of syntax in mathematics (as evidenced by the examples just given), and there are many logical systems (or logics, for short). One of the most important is first-order logic[1], which is the system we will study in this course.

Like any logic, first-order logic has a proof theory and a model theory. Proof theory is focused on syntax: systems for formal reasoning in the logic, and their properties. On the other hand, model theory is focused on semantics. Our objects of study are elementary classes (classes of mathematical structures which can be axiomatized by theories in first-order logic), and relationships between properties of a first-order theory, its elementary class of models, and definability within these models.

Here, "mathematical structure" has a precise meaning: a structure is a set (or a family of sets), equipped with distinguished elements, operations, and relations. From the names for these distinguished elements, operations, and relations, we build up formulas of first-order logic (syntax), which correspond to new definable sets (and elements, operations, relations) in the structure (semantics). For example, the distinguished operations in an algebraically closed

[1] Why is first-order logic important? For practical reasons: it is fairly expressive, while also exhibiting a number of nice properties, like compactness and a good proof system, which we will discuss in this class. And for foundational reasons: ZFC set theory, expressed in first-order logic, is the standard foundation for mathematics (but we will *not* discuss any foundational issues in this class).

field include addition and multiplication; the class of algebraically closed fields is the class of models of a first-order theory ACF, and definable sets relative to this theory are (Boolean combinations of) affine algebraic sets.

So model theory is very abstract, in that we study first-order theories and elementary classes in general, rather than any particular theory or class. But we will often ground ourselves and obtain applications by specializing the general theory to particular examples.

In the first part of this course, we will define the language of first-order logic one piece at a time, looking first at a piece of syntax, and then the corresponding bit of semantics, as shown in the table below. Some of will probably seem pedantic, since we will work at a high level of abstraction, but at the same time the syntax is usually well-chosen to make the intended semantics seem clear. But we will conclude this part with a very non-trivial theorem: Gödel's Completeness Theorem,[2] which shows that the syntactic notion of provability in first-order logic is equivalent to the semantic notation of logical entailment. This is the only bit of proof theory we will do in this course; as an immediate consequence, we get the purely model-theoretic Compactness Theorem, which will be one of our main tools going forward.

| **Syntax** | **Semantics** |
|---|---|
| Vocabularies | Structures |
| Terms | Evaluation |
| Formulas | Satisfaction |
| Theories | Models |
| Provability ($\vdash$) | Entailment ($\models$) |

## 1.1 Vocabularies

A first-order **vocabulary** $\mathcal{V}$ consists of:

1. A nonempty set $\mathcal{S}$ of **sorts**.

2. A set $\mathcal{F}$ of **function symbols**.[3] Each function symbol $f \in \mathcal{F}$ has a **type** $(s_1, \ldots, s_n) \to s$, where $\{s_1, \ldots, s_n, s\} \subseteq \mathcal{S}$. In the case $n = 0$, we call the function symbol a **constant symbol** of type $s$.

3. A set $\mathcal{R}$ of **relation symbols**. Each relation symbol $R \in \mathcal{R}$ has a **type** $(s_1, \ldots, s_n)$, where $\{s_1, \ldots, s_n\} \subseteq S$. In the case $n = 0$, we call the relation symbol a **proposition symbol**.

[2]Not to be confused with Gödel's Incompleteness Theorem!

[3]We are purposefully vague about what counts as a symbol (or a sort, for that matter); the name symbol is meant to suggest something that you could write down with a pencil on paper, but we have no intention of formalizing this notion! In practice, real-world symbols on paper can be encoded as mathematical objects (e.g. sets) in any way you like, and a symbol can be any mathematical object. In particular, a vocabulary may be uncountably infinite.

In the case that $\mathcal{S}$ is a singleton $\{s\}$, we say that $\mathcal{V}$ is **single-sorted**.[4] In this case, we typically do not name the single sort. The type of a function symbol $f \in \mathcal{F}$ is always $(s, \dots, s) \to s$, and the length of the tuple $(s, \dots, s)$ is called the **arity** of $f$. Similarly, the type of a relation symbol $R \in \mathcal{R}$ is always $(s, \dots, s)$, and the length of the tuple $(s, \dots, s)$ is called the **arity** of $R$. The words **unary**, **binary**, and **ternary** mean arity 1, 2, and 3, respectively, and we also write $n$**-ary** to mean arity $n$ for $n \neq 1, 2, 3$. In the single-sorted setting, constant symbols are 0-ary function symbols, and proposition symbols are 0-ary relation symbols.

When $\mathcal{V}$ is finite, it is often convenient to list it as

$$(s_1, \dots, s_l; f_1, \dots, f_m; R_1, \dots, R_n).$$

It is to be understood from the notation that $\mathcal{S} = \{s_1, \dots, s_l\}$, $\mathcal{F} = \{f_1, \dots, f_m\}$, and $\mathcal{R} = \{R_1, \dots, R_n\}$. The types of the function and relation symbols are suppressed. In the single-sorted case, the single sort is usually omitted.

**Example 1.1.** We begin with an example which demonstrates the use of multiple sorts. The vocabulary of vector spaces is:

$$(k, v; 0_k, 1_k, +_k, \times_k, -_k, 0_v, +_v, -_v, \cdot).$$

- $0_k$ and $1_k$ are constant symbols of type $k$.
- $+_k$ and $\times_k$ are function symbols of type $(k, k) \to k$.
- $-_k$ is a function symbol of type $k \to k$.
- $0_v$ is a constant symbol of type $v$.
- $+_v$ is a function symbol of type $(v, v) \to v$.
- $-_v$ is a function symbol of type $v \to v$.
- $\cdot$ is a function symbol of type $(k, v) \to v$.

This vocabulary can be extended in many ways, e.g. to the vocabulary of inner product spaces, obtained by adding a function symbol $\langle -, - \rangle$ of type $(v, v) \to k$.

**Example 1.2.** There are often multiple vocabularies which are appropriate for a given kind of structure. Which one to use depends on the context. For example, if we are only interested in vector spaces over a fixed field $K$, it may be more convenient to use the vocabulary of $K$-vector spaces. This vocabulary is single-sorted:

$$(0, +, -, (a)_{a \in K}).$$

- 0 is a constant symbol.

[4]Classically, model theory was only concerned with single-sorted vocabularies, but there are advantages to developing the foundations in a multi-sorted setting.

- $+$ is a binary function symbol.
- $-$ is a unary function symbol.
- $a$ is a unary function symbol, for every element $a \in K$.

**Example 1.3.** The vocabulary of graphs is single-sorted, with a single binary relation symbol $E$. The vocabulary of orders is single-sorted with a single binary relation symbol $\leq$. Of course, these vocabularies only differ in the name we've chosen for the unique binary relation symbol; we try to choose symbols which are suggestive of the semantics we have in mind.

**Example 1.4.** We can also mix function symbols and relation symbols in the same vocabulary. The vocabulary of ordered groups is single-sorted:

$$(e, \cdot, ^{-1}, \leq).$$

Here $e$ is a constant symbol, $\cdot$ is a binary function symbol, $^{-1}$ is a unary function symbol, and $\leq$ is a binary relation symbol.

## 1.2 $\mathcal{S}$-indexed sets

Given a nonempty set $\mathcal{S}$, an **$\mathcal{S}$-indexed set** is a family of sets $(A_s)_{s\in\mathcal{S}}$.

If $A = (A_s)_{s\in\mathcal{S}}$ and $B = (B_s)_{s\in\mathcal{S}}$ are $\mathcal{S}$-indexed sets, an **$\mathcal{S}$-indexed map** $f\colon A \to B$ is a family of maps $(f_s\colon A_s \to B_s)_{s\in\mathcal{S}}$. We say that $f$ is **injective** if each component $f_s$ is injective, and **surjective** if each component $f_s$ is surjective. We denote the set of $\mathcal{S}$-indexed maps $A \to B$ by $B^A$.

For each tuple $(s_1, \ldots, s_n)$ from $\mathcal{S}$, we define $A_{(s_1,\ldots,s_n)} = \prod_{i=1}^n A_{s_i}$. In the case $n = 0$, we have the empty product, which is a singleton set $A_{()} = \{*\}$. An $\mathcal{S}$-indexed map $f\colon A \to B$ induces a map $f_{(s_1,\ldots,s_n)}\colon A_{(s_1,\ldots,s_n)} \to B_{(s_1,\ldots,s_n)}$ by

$$f_{(s_1,\ldots,s_n)}(a_1, \ldots, a_n) = (f_{s_1}(a_1), \ldots, f_{s_n}(a_n)).$$

Let $A = (A_s)_{s\in\mathcal{S}}$ be an $\mathcal{S}$-indexed set. The **cardinality** of $A$, denoted $|A|$, is just the cardinality of the disjoint union $\bigsqcup_{s\in\mathcal{S}} A_s$.

All of the standard operations on sets can be extended to $\mathcal{S}$-indexed sets, componentwise. For example, $A \cup B = (A_s \cup B_s)_{s\in\mathcal{S}}$, $A \cap B = (A_s \cap B_s)_{s\in\mathcal{S}}$, $A \setminus B = (A_s \setminus B_s)_{s\in\mathcal{S}}$, and $A \times B = (A_s \times B_s)_{s\in\mathcal{S}}$. We write $A \subseteq B$ if $A_s \subseteq B_s$ for all $s \in \mathcal{S}$.

## 1.3 Structures

Let $\mathcal{V} = (\mathcal{S}; \mathcal{F}; \mathcal{R})$ be a vocabulary. A $\mathcal{V}$-structure $A$ consists of:

1. An $\mathcal{S}$-indexed set $(A_s)_{s\in\mathcal{S}}$, called the **domain** of $A$, which we also denote by $A$.[5]

[5] Contrary to the traditional approach, we allow empty sorts and empty structures: In an $\mathcal{V}$-structure $A$, we may have $A_s = \emptyset$ for some sort $s$, or even for every sort $s$.

2. A function $f^A\colon A_{(s_1,\dots,s_n)} \to A_s$, called the **interpretation** of $f$ in $A$, for each function symbol $f \in \mathcal{F}$ of type $(s_1,\dots,s_n) \to s$. In the case of a constant symbol $c$ of type $s$, we have $c^A\colon \{*\} \to A_s$, and we identify $c^A$ with $c^A(*) \in A_s$.

3. A set $R^A \subseteq A_{(s_1,\dots,s_n)}$, called the **interpretation** of $R$ in $A$, for each relation symbol $R \in \mathcal{R}$ of type $(s_1,\dots,s_n)$. In the case of a proposition symbol $P$, we have $P^A \subseteq \{*\}$, and $P^A$ is either $\{*\}$ ("true", or 1) or $\emptyset$ ("false", or 0).

If $\mathcal{V}$ is listed as $(s_1,\dots,s_l; f_1,\dots,f_n; R_1,\dots,R_m)$, then we often present a structure $A$ as

$$(A_{s_1},\dots,A_{s_l}; f_1^A,\dots,f_n^A; R_1^A,\dots,R_m^A).$$

If $A$ and $B$ are $\mathcal{V}$-structures, a **homomorphism** $h\colon A \to B$ is an $\mathcal{S}$-indexed map such that:

1. For every function symbol $f \in \mathcal{F}$ of type $(s_1,\dots,s_n) \to s$, and for every tuple $a = (a_1,\dots,a_n) \in A_{(s_1,\dots,s_n)}$,

$$h_s(f^A(a)) = f^B(h_{(s_1,\dots,s_n)}(a)).$$

2. For every relation symbol $R \in \mathcal{R}$ of type $(s_1,\dots,s_n)$, and for every tuple $a = (a_1,\dots,a_n) \in A_{(s_1,\dots,s_n)}$,

$$a \in R^A \implies h_{(s_1,\dots,s_n)}(a) \in R^B.$$

(We say $h$ **preserves** $R$.)

A homomorphism $h\colon A \to B$ is an **embedding** if additionally it is injective and for every relation symbol $R \in \mathcal{R}$ of type $(s_1,\dots,s_n)$, and for every tuple $a = (a_1,\dots,a_n) \in A_{(s_1,\dots,s_n)}$,

$$a \in R^A \iff h_{(s_1,\dots,s_n)}(a) \in R^B.$$

(We say $h$ **preserves and reflects** $R$.)

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Suppose $A$ and $B$ are $\mathcal{V}$-structures and $A \subseteq B$. Then $A$ is a **substructure** of $B$ if the inclusion map $A \to B$ is an embedding. That is, $f^A(a) = f^B(a)$ for all $a \in A_{(s_1,\dots,s_n)}$ and all $f \in \mathcal{F}$ of type $(s_1,\dots,s_n) \to s$, and $a \in R^A$ iff $a \in R^B$ for all $a \in A_{(s_1,\dots,s_n)}$ and $R \in \mathcal{R}$ of type $(s_1,\dots,s_n)$.

Suppose $A$ is an $\mathcal{S}$-indexed subset of a $\mathcal{V}$-structure $B$. We say that $A$ is **$\mathcal{V}$-closed** if $A$ is closed under the functions $f^B$ for all $f \in \mathcal{F}$: if $f$ has type $(s_1,\dots,s_n) \to s$, then for all $(a_1,\dots,a_n) \in A_{(s_1,\dots,s_n)}$, we have $f^B(a_1,\dots,a_n) \in A_s$.

If $A$ is $\mathcal{V}$-closed, then there is a unique substructure of $B$ with domain $A$, called the **induced substructure** on $A$, defined by:

$$f^A(a) = f^B(a), \text{ for all } f \in \mathcal{F} \text{ of type } (s_1,\dots,s_n) \to s \text{ and } a \in A_{(s_1,\dots,s_n)}.$$
$$R^A(a) \iff R^B(a), \text{ for all } R \in \mathcal{R} \text{ of type } (s_1,\dots,s_n) \text{ and } a \in A_{(s_1,\dots,s_n)}.$$

If $A$ is an arbitrary $\mathcal{S}$-indexed subset of a $\mathcal{V}$-structure $B$, then the induced substructure on the $\mathcal{V}$-closure of $A$ is the smallest substructure of $B$ containing $A$. This is called the **substructure generated by** $A$ and denoted $\langle A \rangle$.

An **isomorphism** is a homomorphism $h\colon A \to B$ such that there exists an inverse homomorphism $h^{-1}\colon B \to A$. Equivalently, $h$ is a surjective embedding. We write $A \cong B$ when $A$ and $B$ are isomorphic.

An **automorphism** of $A$ is an isomorphism $\sigma\colon A \to A$. The automorphisms of $A$ form a group under composition, denoted $\mathrm{Aut}(A)$. If $C \subseteq A$, we denote by $\mathrm{Aut}(A/C)$ the subgroup of automorphisms of $A$ fixing $C$ pointwise:

$$\mathrm{Aut}(A/C) = \{\sigma \in \mathrm{Aut}(A) \mid \sigma_s(c) = c \text{ for all } c \in C_s\}.$$

Later, we will meet other kinds of maps of interest between structures.

## 1.4 Terms and evaluation

Given the set $\mathcal{S}$ of sorts, we introduce an $\mathcal{S}$-indexed set $X = (X_s)_{s\in\mathcal{S}}$ of **variables**. We will assume that the sets $X_s$ are pairwise disjoint and countably infinite. The actual identity of the variables will not matter to us, so we will feel free to use different names for them in different situations; the important thing is that each variable has a fixed type $s$, and when we work with finitely many variables at a time, we never run out of variables of any type.

A **variable context** is a finite tuple $x = (x_1, \dots, x_k)$ of variables. If $s_i$ is the type of $x_i$, we say the context has type $(s_1, \dots, s_k)$. Note that there is an empty variable context $()$. We will abuse notation freely when concatenating variable contexts. For example, when $x = (x_1, \dots, x_k)$ is a variable context and $y$ is another variable not in $x$, we write $xy$ or $x, y$ for the variable context $(x_1, \dots, x_k, y)$.

A **$\mathcal{V}$-term** of type $s \in \mathcal{S}$ in context $x = (x_1, \dots, x_k)$ is one of the following:

- A variable $x_i$ of type $s$.
- A constant symbol $c \in \mathcal{F}$ of type $s$.
- A **composite term** $f(t_1, \dots, t_n)$, where $f \in \mathcal{F}$ is a function symbol of type $(s_1, \dots, s_n) \to s$ and $t_i$ is a $\mathcal{V}$-term of type $s_i$ in context $x$, for all $1 \leq i \leq n$.

Note that the case of constant symbols is really a special case of the case of composite terms, when $n = 0$.

This is a definition by recursion (simultaneously across all types $s$), so we obtain a corresponding method of proof by induction. To prove a claim about all $\mathcal{V}$-terms in context $x$, it suffices to check the base case (the claim holds for all variables in $x$), and the inductive step (given that the claim holds for the terms $t_1, \dots, t_n$, it holds for the composite term $f(t_1, \dots, t_n)$). Sometimes it is useful to handle the constant symbols as a separate base case.

We denote by $\mathcal{T}_s(x)$ the set of $\mathcal{V}$-terms of type $s$ in context $x$, and by $\mathcal{T}(x) = (\mathcal{T}_s(x))_{s\in\mathcal{S}}$ the $\mathcal{S}$-indexed set of all $\mathcal{V}$-terms in context $x$. We usually write $t(x)$ to denote that the term $t$ is in context $x$.

**Example 1.5.** In the vocabulary of vector spaces defined in Example 1.1, the following are terms in context $(x, y)$, where $x$ has type $k$ and $y$ has type $v$:

$$\begin{array}{lllll} \text{Type } k\colon & 1_k, & x, & (x +_k 0_k) \times_k x, & -_k(x) +_k x \\ \text{Type } v\colon & 0_v, & y, & (x \times_k x) \cdot (y +_v y), & -_v(-_v(-_v(y))) \end{array}$$

Note that we the natural notation for our symbols when they differ from the formal syntax described above, for example writing $(x +_k 0_k)$ instead of $+_k(x, 0_k)$.

Given a variable context $x = (x_1, \dots, x_k)$ of type $(s_1, \dots, s_k)$ and an $\mathcal{S}$-indexed set $A$, an **interpretation** of $x$ in $A$ is a function $I\colon \{x_1, \dots, x_k\} \to A$ such that $I(x_i) \in A_{s_i}$ for all $i$.[6] Of course, there is a unique interpretation of the empty context $()$, namely the empty function. The set of all interpretations of $x$ in $A$ is denoted $A^x$. We usually identify an interpretation $x_i \mapsto a_i$ with the image tuple $a = (a_1, \dots, a_k)$. Note that this identification describes a bijection $A^x \cong A_{(s_1,\dots,s_k)}$.

Let $A$ be a $\mathcal{V}$-structure, and let $a = (a_1, \dots, a_k)$ be an interpretation of $x = (x_1, \dots, x_k)$ in $A$. Then there is an $\mathcal{S}$-indexed map $\mathrm{eval}_a\colon \mathcal{T}(x) \to A$ defined by recursion:

- $\mathrm{eval}_a(x_i) = a_i$.
- $\mathrm{eval}_a(c) = c^A$.
- $\mathrm{eval}_a(f(t_1, \dots, t_n)) = f^A(\mathrm{eval}_a(t_1), \dots, \mathrm{eval}_a(t_n))$.

We use the notation $t^A(a)$ for $\mathrm{eval}_a(t(x))$.

Instead of fixing the interpretation $a$ and letting the term $t(x)$ vary, we can fix the term $t(x)$ and let the interpretation $a$ vary. Then $t(x)$ determines a function $t^A\colon A^x \to A_s$, by $a \mapsto t^A(a)$.

**Remark 1.6.** If $t(x)$ is a term in context $x$, and $y$ is a variable not in $x$, then $t$ is also a term in context $xy$. Indeed, the context just restricts which variables can be mentioned in $t$. When we want to consider $t$ in context $xy$, we write $t(x, y)$. Note that if $t(x)$ is a term in context $x$, $a$ is an interpretation of $x$ in $A$ and $b$ is any interpretation of $y$ in $A$, then $t^A(a) = t^A(a, b)$.

**Exercise 1.** Let $B$ be a $\mathcal{V}$-structure, and $A \subseteq B$. Show that $b \in \langle A \rangle$ if and only if there is a variable context $x$, a term $t(x)$, and an interpretation $a \in A^x$ such that $b = t^A(a)$.

**Exercise 2.** Explain what the following statement means and prove it: $\mathcal{T}(x)$ is the free $\mathcal{V}$-structure on generators $x$. (This includes explaining how to make $\mathcal{T}(x)$ into a $\mathcal{V}$-structure. How do you interpret relation symbols in $\mathcal{T}(x)$?)

[6] If we defined variable contexts as $\mathcal{S}$-indexed sets instead of as tuples, this would the the same as an $\mathcal{S}$-indexed map $x \to A$. We maintain the view of variable contexts as tuples instead of sets in order to not stray too far from traditional notation. The notational distinction between tuples and the sets they enumerate is frequently abused in model theory.

## 1.5 Formulas and satisfaction

An **atomic $\mathcal{V}$-formula** in context $x$ is one of the following:

- $(t_1 = t_2)$, where $t_1$ and $t_2$ are $\mathcal{V}$-terms of type $s$ in context $x$, for some $s \in S$.
- $R(t_1, \ldots, t_n)$, where $R \in \mathcal{R}$ is a relation symbol of type $(s_1, \ldots, s_n)$ and $t_i$ is a $\mathcal{V}$-term of type $s_i$ in context $x$, for all $1 \leq i \leq n$.

A **$\mathcal{V}$-formula** in context $x$ is one of the following:

- An atomic $\mathcal{V}$-formula in context $x$.
- $\top$ or $\bot$.
- $(\psi \wedge \chi)$, $(\psi \vee \chi)$, or $\neg\psi$, where $\psi$ and $\chi$ are $\mathcal{V}$-formulas in context $x$.
- $\exists y\, \psi$, where $y$ is a variable not in $x$ and $\psi$ is a $\mathcal{V}$-formula in context $xy$.

This is a definition by recursion (simultaneously across all contexts $x$), so we obtain a corresponding method of proof by induction. To prove a claim about all $\mathcal{V}$-formulas, it suffices to check the base case (the claim holds for all atomic formulas, $\top$, and $\bot$), and the inductive steps (given that the claim holds for the $\mathcal{V}$-formulas $\psi$ and $\chi$, it holds for the formulas $(\psi \wedge \chi)$, $(\psi \vee \chi)$, $\neg\psi$, and $\exists y\, \psi$).

We say that a variable $y$ is **bound** in $\varphi$ if some subformula of $\varphi$ has the form $\exists y\, \psi$. Note that according to our definitions, if $\varphi$ is in context $x$, then no variable in $x$ can be bound in $\varphi$.

We will also employ the following standard shorthands:

- $(t_1 \neq t_2)$ is shorthand for $\neg(t_1 = t_2)$.
- $(\psi \to \chi)$ is shorthand for $(\neg\psi \vee \chi)$.
- $(\psi \leftrightarrow \chi)$ is shorthand for $((\psi \to \chi) \wedge (\chi \to \psi))$.
- $\bigwedge_{i=1}^n \varphi_i$ and $\bigvee_{i=1}^n \varphi_i$ are shorthand for $(\ldots((\varphi_1 \wedge \varphi_2) \wedge \varphi_3) \cdots \wedge \varphi_n)$ and $(\ldots((\varphi_1 \vee \varphi_2) \vee \varphi_3) \cdots \vee \varphi_n)$, respectively. In the case $n = 0$, the empty conjunction is $\top$ and the empty disjunction is $\bot$.
- $\forall y\, \psi$ is shorthand for $\neg\exists y\, \neg\psi$.[7]

We denote by $\mathcal{L}_x$ the set of all $\mathcal{V}$-formulas in context $x$, and we usually write $\varphi(x)$ to denote that the formula $\varphi$ is in context $x$. We denote by $\mathcal{L}$ the set of all $\mathcal{V}$-formulas, which is also called the first-order **language** corresponding to vocabulary $\mathcal{V}$. When there is possibility for confusion, we can make the vocabulary explicit by writing $\mathcal{L}(\mathcal{V})$.

[7]It may seem perverse to define the universal quantifier in terms of the existential quantifier, but doing this has the advantage that we have fewer cases to check in proofs by induction on formulas. Of course, which logical connectives we take as primitive is a matter of convention. We could have similarly taken $(\psi \wedge \chi)$ as shorthand for $\neg(\neg\psi \vee \neg\chi)$, or done away with $\wedge$, $\vee$, and $\neg$ altogether in favor of the Sheffer stroke $\uparrow$.

It is notationally common to begin a discussion by fixing a first-order language $\mathcal{L}$, which is implicitly built from a vocabulary, which remains anonymous. Then we write things like $\mathcal{L}$-structure and $\mathcal{L}$-formula in place of $\mathcal{V}$-structure and $\mathcal{V}$-formula, where $\mathcal{V}$ is the vocabulary of $\mathcal{L}$.[8]

**Example 1.7.** In the vocabulary of ordered groups defined in Example 1.4, the following are formulas:

$$\begin{aligned} \text{Context } (x,y,z)\colon & (x \cdot y = e), \quad (x \leq y \cdot z), \quad \exists w\, (x \cdot w \leq z \wedge z \leq y \cdot w) \\ \text{Empty context } ()\colon & \forall x\, x \cdot x^{-1} = e, \quad \exists y\, (y \neq e \wedge \forall x\, x \cdot y = y \cdot x) \end{aligned}$$

Note that we the natural notation for our symbols when they differ from the formal syntax described above, for example writing $x \leq y$ instead of $\leq(x, y)$ and $x^{-1}$ instead of $^{-1}(x)$.

Let $A$ be a $\mathcal{V}$-structure, let $\varphi$ be a $\mathcal{V}$-formula in context $x$, and let $a$ be an interpretation of $x$ in $A$. We define the relation $A \models \varphi(a)$, read $A$ **satisfies** $\varphi(a)$ or $\varphi(a)$ **is true in** $A$, by induction on the structure of $\varphi$:

- If $\varphi$ is $(t_1 = t_2)$, then $A \models \varphi(a)$ iff $t_1^A(a) = t_2^A(a)$.
- If $\varphi$ is $R(t_1, \ldots, t_n)$, then $A \models \varphi(a)$ iff $(t_1^A(a), \ldots, t_n^A(a)) \in R^A$.
- If $\varphi$ is $\top$, then $A \models \varphi(a)$. If $\varphi$ is $\bot$, then $A \not\models \varphi(a)$.
- If $\varphi$ is $(\psi \wedge \chi)$, then $A \models \varphi(a)$ iff $A \models \psi(a)$ and $A \models \chi(a)$.
- If $\varphi$ is $(\psi \vee \chi)$, then $A \models \varphi(a)$ iff $A \models \psi(a)$ or $A \models \chi(a)$.
- If $\varphi$ is $\neg\psi$, then $A \models \varphi(a)$ iff $A \not\models \psi(a)$.
- If $\varphi$ is $\exists y\, \psi$, where $y$ has type $s$, then $A \models \varphi(a)$ iff there exists some $b \in A_s$ such that $A \models \psi(a, b)$.

As a consequence of these definitions, our shorthands also have their expected meanings. For example:

- If $\varphi$ is $\psi \to \chi$, then $A \models \varphi$ iff $A \not\models \psi$ or $A \models \chi$, i.e. if $A \models \psi$, then $A \models \chi$.
- If $\varphi$ is $\forall y\, \psi$, where $y$ has type $s$, then $A \models \varphi$ iff there does *not* exist $b \in A_s$ such that $A \not\models \psi(a, b)$, i.e. for all $b \in A_s$, $A \models \psi(a, b)$.

**Remark 1.8.** As in Remark 1.6, if $\varphi(x)$ is a formula in context $x$, and $y$ is a variable not in $x$ and which is not bound in $\varphi$, then $\varphi$ is also a formula in context $xy$. When we want to consider $\varphi$ in context $xy$, we write $\varphi(x, y)$. Note that if $\varphi(x)$ is a formula in context $x$, $a$ is an interpretation of $x$ in $A$ and $b$ is any interpretation of $y$ in $A$, then $A \models \varphi(a)$ if and only if $A \models \varphi(a, b)$.

**Exercise 3.** Prove the assertions in Remarks 1.6 and 1.8 by induction on the structure of terms and formulas.

[8] In many sources, the distinction we have made between language and vocabulary is blurred, and only the term language is used.

### 1.6 Theories and models

Tuesday 8/28 A $\mathcal{V}$**-sentence** $\varphi$ is a $\mathcal{V}$-formula in the empty context. When $A$ is a $\mathcal{V}$-structure, there is a unique interpretation of the empty context in $A$. Then we write $A \models \varphi$ or $A \not\models \varphi$, without including the interpretation in the notation.

A $\mathcal{V}$**-theory** $T$ is a set of $\mathcal{V}$-sentences. We write $A \models T$, read $A$ **satisfies** $T$ or $A$ **is a model of** $T$, if $A \models \varphi$ for all $\varphi \in T$.

Let $T$ be a theory, and let $\varphi$ be a sentence. Then we write $T \models \varphi$, read $T$ **entails** $\varphi$, if every model of $T$ satisfies $\varphi$. Of course, if $\varphi \in T$, then $T \models \varphi$ by unraveling the definitions.

A theory $T$ is **complete** if for every sentence $\varphi$, either $T \models \varphi$ or $T \models \neg\varphi$.

A theory $T$ is **satisfiable** if $T$ has a model. Note that $T$ is satisfiable if and only if $T \not\models \bot$. Indeed, if $T$ has a model $A$, then $A \not\models \bot$, so $T \not\models \bot$. On the other hand, if $T$ has no models, then $T \models \bot$ vacuously.

**Example 1.9.** Let $A$ be any $\mathcal{V}$-structure. Then the **complete theory of** $A$ is

$$\mathrm{Th}(A) = \{\varphi \in \mathcal{L}_{()} \mid A \models \varphi\}.$$

The name is justified, since for any sentence $\varphi$, either $A \models \varphi$ or $A \not\models \varphi$. In the first case, $\varphi \in \mathrm{Th}(A)$, while in the second case $A \models \neg\varphi$, so $\neg\varphi \in \mathrm{Th}(A)$.

**Example 1.10.** Let $\mathcal{V} = (e, \cdot, ^{-1})$ be the single-sorted vocabulary of groups. The theory $T_{\mathrm{grp}}$ of groups consists of the following sentences:

$$\forall x \forall y \forall z\, ((x \cdot y) \cdot z = x \cdot (y \cdot z))$$
$$\forall x\, ((x \cdot e = x) \wedge (e \cdot x = x))$$
$$\forall x\, ((x \cdot x^{-1} = e) \wedge (x^{-1} \cdot x = e))$$

Of course, a $\mathcal{V}$-structure is a group if and only if $A \models T_{\mathrm{grp}}$.

The theory $T_{\mathrm{grp}}$ is not complete. For example, if $\varphi$ is the sentence

$$\forall x \forall y\, (x \cdot y = y \cdot x),$$

then $T \not\models \varphi$ and $T \not\models \neg\varphi$, since there are both abelian and non-abelian groups.

On the other hand, we have

$$T_{\mathrm{grp}} \models (\forall x\, (x \cdot x = e) \to \forall x \forall y\, (x \cdot y = y \cdot x)),$$

since all groups of exponent 2 are abelian.

Given a theory $T$, how can we tell when $T \models \varphi$, or even when $T$ is satisfiable? Well, we need to provide a proof. This can be an *external* proof, obtained by "ordinary mathematical" reasoning about the models of $T$ (as in the previous example). How easy this is depends on how well we understand the models of $T$. But it can also be useful to know that that is a robust notion of proof *internal* to first-order logic, obtained by directly manipulating the syntax, without thinking about models at all. Our goal is to define this notion of proof, denoted $\vdash$, and show that provability exactly captures entailment: $T \vdash \varphi$ iff $T \models \varphi$.

As a warm-up, and to serve as firm footing from which to launch into completeness for first-order logic, we will take a detour through the world of Boolean algebras and propositional logic.

# 2 Boolean algebras

## 2.1 Orders, lattices, Boolean algebras

Thursday 8/30 A **preorder** $(X; \preceq)$ is a set $X$ together with a binary relation $\preceq$ on $X$ such that for all $x, y, z \in X$:

- $x \preceq x$.
- If $x \preceq y$ and $y \preceq z$, then $x \preceq z$.

A preorder is a **partial order** if it additionally satisfies:

- If $x \preceq y$ and $y \preceq x$, then $x = y$.

If $(X; \preceq)$ is a preorder, then the relation $x \sim y$ iff $x \preceq y$ and $y \preceq x$ is an equivalence relation on $X$. We denote the equivalence class of $x$ by $[x]$. The quotient $X/\sim$ comes equipped with a relation $\leq$, defined by $[x] \leq [y]$ if and only if $x \preceq y$. Then $(X/\sim; \leq)$ is a partial order.

A **lattice** is a partial order $(X; \leq)$ equipped with distinguished elements $\top$ ("top") and $\bot$ ("bottom") and binary operations $\wedge$ ("meet") and $\vee$ ("join") such that:

- $\top$ is the maximum element, i.e. for all $x$, $x \leq \top$.
- $\bot$ is the minimum element, i.e. for all $x$, $\bot \leq x$.
- $x \wedge y$ is the greatest lower bound of $x$ and $y$, i.e. for all $z$, $z \leq x \wedge y$ if and only if $z \leq x$ and $z \leq y$.
- $x \vee y$ is the least upper bound of $x$ and $y$, i.e. for all $z$, $x \vee y \leq z$ if and only if $x \leq z$ and $y \leq z$.

In a lattice, any finite set $S = \{x_1, \ldots, x_n\}$ has a greatest lower bound:

$$\bigwedge S = (\ldots((x_1 \wedge x_2) \wedge x_3) \cdots \wedge x_n),$$

and a least upper bound:

$$\bigvee S = (\ldots((x_1 \vee x_2) \vee x_3) \cdots \vee x_n).$$

The greatest lower bound of the empty set is $\top$, and the least upper bound of the empty set is $\bot$.

**Example 2.1.** Let $A$ be any set. Then $(\mathcal{P}(A), A, \emptyset, \cap, \cup)$ is a lattice, called the **powerset lattice** on $A$.

Now let $A$ be a $\mathcal{V}$-structure. Let $\mathrm{Sub}(A)$ be the set of all substructures of $A$. Then $\mathrm{Sub}(A)$ forms a lattice, with top element $\top = A$, bottom element $\bot = \langle \emptyset \rangle$, meet $B \wedge C = B \cap C$, and join $B \vee C = \langle B \cup C \rangle$. When $\mathcal{V}$ is single-sorted and contains only relation symbols, then $\mathrm{Sub}(A)$ is isomorphic to the powerset lattice on $A$.

**Exercise 4.** Lattices also have a purely algebraic (or "equational") definition, which doesn't mention the order relation $\leq$. We'll use the term algebraic lattice for this definition, to distinguish it from the previous definition.

An **algebraic lattice** is a set $X$ equipped with distinguished elements $\top$ and $\bot$ and binary operations $\wedge$ and $\vee$ such that:

1. $(X; \wedge, \top)$ is a commutative idempotent monoid: for all $x, y, z \in X$,
   (a) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$.
   (b) $x \wedge \top = \top \wedge x = x$.
   (c) $x \wedge y = y \wedge x$
   (d) $x \wedge x = x$.
2. $(X; \vee, \bot)$ is a commutative idempotent monoid: for all $x, y, z \in X$,
   (a) $(x \vee y) \vee z = x \vee (y \vee z)$.
   (b) $x \vee \bot = \bot \vee x = x$.
   (c) $x \vee y = y \vee x$.
   (d) $x \vee x = x$.
3. $\wedge$ and $\vee$ satisfy the "absorption laws": for all $x, y \in X$,
   (a) $x \vee (x \wedge y) = x$.
   (b) $x \wedge (x \vee y) = x$.

Show that if $(X; \leq, \top, \bot, \wedge, \vee)$ is a lattice, then $(X; \top, \bot, \wedge, \vee)$ is an algebraic lattice. Conversely, show that if $(X; \top, \bot, \wedge, \vee)$ is an algebraic lattice, then for all $x, y \in X$, we have $x \wedge y = x$ iff $x \vee y = y$. And if we define $x \leq y$ iff these equivalent conditions hold, then $(X; \leq, \top, \bot, \wedge, \vee)$ is a lattice.

A **distributive lattice** is a lattice which satisfies the distributive law: for all $x, y, z \in X$,

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

**Exercise 5.** Let $X$ be a lattice. Show that $X$ satisfies the distributive law if and only if it satisfies the dual distributive law: for all $x, y, z \in X$,

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

**Example 2.2.** The powerset lattice on any set is distributive. So is the subalgebra lattice $\mathrm{Sub}(A)$ of any $\mathcal{V}$-structure when the function symbols in $\mathcal{V}$ have arity at most 1, since in this case $B \vee C = \langle B \cup C \rangle = B \cup C$ when $B$ and $C$ are substructures. But when $\mathcal{V}$ contains function symbols of arity at least 2, $\mathrm{Sub}(A)$ is typically not distributive.

If $x$ is an element of a distributive lattice, then a **complement** of $x$ is an element $y$ such that $x \wedge y = \bot$ and $x \vee y = \top$.

A **Boolean algebra** is a distributive lattice equipped with an additional unary operation $\neg$ ("complement"), such that $\neg x$ is a complement of $x$.

It follows from Exercise 4 that we can also give a purely algebraic definition of Boolean algebras, by adding the following axioms to the axioms listed there:

4. The distributive law: for all $x, y, z \in X$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
5. Complements: for all $x \in X$,
   (a) $x \wedge \neg x = \bot$.
   (b) $x \vee \neg x = \top$.

**Example 2.3.** (a) Let $A$ be a set. Then $(\mathcal{P}(A); \subseteq, A, \emptyset, \cap, \cup, -)$ is a Boolean algebra. Here $-B = A \setminus B$.

(b) Let $A$ be an infinite set, and let $\mathcal{P}^*(A) = \{B \subseteq A \mid B \text{ is finite, or cofinite}\}$ ($B$ is cofinite if $A \setminus B$ is finite). Then $(\mathcal{P}^*(A); \subseteq, A, \emptyset, \cap, \cup, -)$ is a Boolean algebra.

(c) Let $X$ be a topological space. Then the family $\mathcal{O}(X)$ of open sets in $X$ is not naturally equipped with a Boolean algebra structure, since open sets are typically not closed under complement. But letting $\mathrm{Cl}(X)$ be the set of clopen sets in $X$, $(\mathrm{Cl}(X); \subseteq, X, \emptyset, \cap, \cup, -)$ does form a Boolean algebra.

Later, we'll see more examples of Boolean algebras which are not algebras of sets under the standard set operations ($\cap$, $\cup$, and $-$). But there is a good reason for giving examples of this form: Stone duality tells us that every Boolean algebra is isomorphic to a subalgebra of a powerset algebra. In fact, every Boolean algebra is isomorphic to the algebra of clopen sets in a topological space!

**Exercise 6.** Let $X$ be a distributive lattice. Show that if $y$ and $z$ are both complements of $x$, then $y = z$. So in a Boolean algebra, $\neg x$ is the unique complement of $x$.

**Exercise 7.** Let $B$ be a Boolean algebra. Show that for all $x, y \in B$,

(a) $\neg\neg x = x$.

(b) (De Morgan's Laws) $\neg(x \wedge y) = (\neg x \vee \neg y)$ and $\neg(x \vee y) = (\neg x \wedge \neg y)$.

(c) $\neg\top = \bot$ and $\neg\bot = \top$.

(d) $x \leq y$ iff $\neg y \leq \neg x$.

**Exercise 8.** Let $B$ be a Boolean algebra. For any $x, y \in B$, we define

$$\begin{aligned} x \to y &= \neg x \vee y \\ x \leftrightarrow y &= (x \to y) \wedge (y \to x). \end{aligned}$$

Show that $x \leq y$ iff $x \to y = \top$, and $x = y$ iff $x \leftrightarrow y = \top$.

## 2.2 Filters and ultrafilters

In this subsection, we fix a Boolean algebra $B$. A subset $F \subseteq B$ is a **filter** if:

1. $F$ is closed upwards: If $x \in F$ and $x \leq y$, then $y \in F$.
2. $F$ is closed under finite meets: $\top \in F$, and if $x, y \in F$, then $x \wedge y \in F$.

A filter $F$ is **proper** if $\bot \notin F$ (equivalently, $F \subsetneq B$).

Dually, a subset $I \subseteq B$ is a **ideal** if:

1. $I$ is closed downwards: If $x \in I$ and $y \leq x$, then $y \in I$.
2. $I$ is closed under finite joints: $\bot \in I$, and if $x, y \in I$, then $x \vee y \in I$.

It follows from Exercise 7 that if $F$ is a filter, then $\neg F = \{\neg x \mid x \in F\}$ is an ideal, and vice versa.

Let $A$ be an arbitrary subset of $B$. Then we define:

$$\mathrm{Fil}(A) = \{x \in B \mid \bigwedge A' \leq x \text{ where } A' \text{ is a finite subset of } A\}.$$

We call $\mathrm{Fil}(A)$ the **filter generated by** $A$, a name which is justified by the following lemma.

**Lemma 2.4.** $\mathrm{Fil}(A)$ *is the smallest filter containing* $A$.

*Proof.* If $F$ is a filter containing $A$, then since $F$ is closed under finite meets, $\bigwedge A' \in F$ for any finite $A' \subseteq A$. And since $F$ is closed upwards, $x \in F$ for any $x \in \mathrm{Fil}(A)$. So $\mathrm{Fil}(A) \subseteq F$. It remains to show that $\mathrm{Fil}(A)$ is a filter.

Closure upwards: If $x \in \mathrm{Fil}(A)$, and $x \leq y$, then $\bigwedge A' \leq x \leq y$ for some finite $A' \subseteq A$, so $y \in \mathrm{Fil}(A)$.

Closure under finite meets: $\top \in \mathrm{Fil}(A)$, since $\top = \bigwedge \emptyset$. And if $x, y \in F$, witnessed by $\bigwedge A_1 \leq x$ and $\bigwedge A_2 \leq y$ for finite $A_1, A_2 \subseteq A$, then

$$\bigwedge (A_1 \cup A_2) = \bigwedge A_1 \wedge \bigwedge A_2 \leq x \wedge y,$$

so $x \wedge y \in \mathrm{Fil}(A)$. $\square$

An **ultrafilter** is a proper filter $U$ such that $x \in U$ or $\neg x \in U$ for all $x \in B$.

**Exercise 9.** Let $B$ be a Boolean algebra.

1. If $f \colon B \to B'$ is a Boolean algebra homomorphism, then $f^{-1}(\{\top^{B'}\}) \subseteq B$ is a filter. We say that $F$ is the **kernel** of $f$.
2. Conversely, if $F$ is a filter, then there is Boolean algebra $B/F$ and a surjective homomorphism $\pi_F \colon B \to B/F$ such that $F$ is the kernel of $\pi_F$. We call $B/F$ the **quotient of** $B$ **by** $F$. *Hint:* Define an equivalence relation $\sim_F$ on $B$ by

$$x \sim_F y \text{ iff } (x \leftrightarrow y) \in F.$$

3. Show that the set of ultrafilters on $B$ is in bijection with the set of homomorphisms $B \to 2$, where $2$ is the two-element Boolean algebra $\{\top, \bot\}$.

Tuesday 9/4

**Lemma 2.5.** *For all $x, y \in B$, $x \wedge y = \bot$ if and only if $x \leq \neg y$.*

*Proof.* By Exercises 8 and 7,

$$\begin{aligned} x \leq \neg y \quad &\text{iff} \quad x \to \neg y = \top \\ &\text{iff} \quad \neg x \vee \neg y = \top \\ &\text{iff} \quad x \wedge y = \bot. \end{aligned}$$ □

**Lemma 2.6** (Ultrafilter Lemma)**.** *Every proper filter is contained in an ultrafilter.*

*Proof.* Let $F$ be a proper filter. Consider the poset $(\mathcal{F}, \subseteq)$, where $\mathcal{F}$ is the set of all proper filters containing $F$. It is easy to check that the union of a chain of proper filters is a filter, so by Zorn's Lemma[9], $\mathcal{F}$ contains a maximal element $U$. We will show that $U$ is an ultrafilter.

Suppose $x \in B$ such that $x \notin U$. We'd like to show that $\neg x \in U$. Consider the filter $F' = \mathrm{Fil}(U \cup \{x\})$. We have $F \subseteq U \subsetneq F'$, so by maximality $F'$ is not proper, i.e. $\bot \in F'$. By definition of $F'$, there is some finite $A \subseteq (U \cup \{x\})$ such that $\bigwedge A = \bot$. We may assume $x \in A$, so we may write $A = A' \cup \{x\}$, where $A' \subseteq U$, and $u = \bigwedge A' \in U$. So we have $u \wedge x = \bot$. By Lemma 2.5, $u \leq \neg x$, so $\neg x \in U$. □

**Example 2.7.** In the case that $A = \{x\}$ is a singleton, we have

$$\mathrm{Fil}(A) = \uparrow(x) = \{y \in B \mid x \leq y\}.$$

We call $\uparrow(x)$ the **principal filter generated by** $x$.

An **atom** is a minimal nonbottom element, i.e. an element $x \in B$ such that $x \neq \bot$ but if $y < x$, then $y = \bot$. The atoms in the powerset algebra $\mathcal{P}(A)$ for nonempty $A$ are exactly the singleton sets $\{a\}$. But not every Boolean algebra has atoms.

**Lemma 2.8.** *The principal filter $\uparrow(x)$ is an ultrafilter if and only if $x$ is an atom.*

*Proof.* Suppose $x$ is an atom. Then for all $y \in B$, we have $x = x \wedge (y \vee \neg y) = (x \wedge y) \vee (x \wedge \neg y)$. Since both disjuncts are below $x$, both are either $\bot$ or $x$. And they are not both $\bot$, since $x \neq \bot$. So we have either $x \wedge y = x$, in which case $x \leq y$ and $y \in \uparrow(x)$, or $x \wedge \neg y = x$, in which case $x \leq \neg y$, and $\neg y \in \uparrow(x)$.

Conversely, suppose $\uparrow(x)$ is an ultrafilter. Then $x \neq \bot$, since $\uparrow(x)$ is proper. Suppose $y < x$. Then $y \notin \uparrow(x)$, so $\neg y \in \uparrow(x)$, and $x \leq \neg y$, so $y = y \wedge x = \bot$. □

[9] For those who are squeamish about the axiom of choice, don't be! We will use it freely in this class. But if you're interested in foundational matters, you can take some comfort in the fact that when $B$ is a countable Boolean algebra, no choice is necessary to extend a proper filter $F$ to an ultrafilter. Simply enumerate $B = \{b_i \mid i \in \omega\}$, and build a sequence of filters $F_i$ by induction. Take $F_0 = F$, and define $F_{i+1} = F_i$ if $b_i \in F_i$ or $\neg b_i \in F_i$, and $F_{i+1} = \mathrm{Fil}(F_i \cup \{b_i\})$ otherwise. Now you may check that $\bigcup_{i \in \omega} F_i$ is an ultrafilter.

**Example 2.9.** When the Boolean algebra $B$ is a powerset algebra $\mathcal{P}(A)$, we abuse terminology by calling a filter $U \subseteq \mathcal{P}(A)$ a **filter on** $A$.

By Lemma 2.8, the principal filter $\uparrow(X) = \{Y \subseteq A \mid X \subseteq Y\}$ for $X \in \mathcal{P}(A)$ is an ultrafilter if and only if $|X| = 1$. In this case, we call

$$\uparrow(\{a\}) = \{Y \subseteq A \mid a \in Y\}$$

the **principal ultrafilter generated by** $a$.

When $A$ is finite, every ultrafilter on $A$ is principal. But when $A$ is infinite, there are non-principal ultrafilters on $A$. To see, this, define

$$C = \{X \subseteq A \mid A \setminus X \text{ is finite}\}.$$

This is called the **cofinite filter** or **Fréchet filter** on $A$. When $A$ is infinite, $C$ is a proper filter, and by Lemma 2.6, $C$ extends to an ultrafilter on $A$. But $C$ is not contained in the principal ultrafilter generated by $a$ for any $a \in A$, since $-\{a\} \in C$. It should be noted that this proof really relies on the axiom of choice: it is impossible to write down any explicit example of a non-principal ultrafilter on an infinite set $A$.

The situation is different for other Boolean algebras, of course. Recall the Boolean algebra $\mathcal{P}^*(A)$ from Example 2.3. In this algebra, the cofinite filter $C$ (defined just as above) is already an ultrafilter.

## 2.3 Stone duality

Let $B$ be a Boolean algebra. We define

$$S(B) = \{U \mid U \subseteq B \text{ is an ultrafilter}\}.$$

For every element $x \in B$, let $[x] = \{U \in S(B) \mid x \in U\}$. So $U \in [x]$ iff $x \in U$. This can be a bit confusing!

**Lemma 2.10.** *For all $x, y \in B$,*

*(1) If $x \leq y$, then $[x] \subseteq [y]$.*

*(2) $[\top] = S(B)$ and $[\bot] = \emptyset$.*

*(3) $[\neg x] = S(B) \setminus [x]$.*

*(4) $[x \wedge y] = [x] \cap [y]$.*

*(5) $[x \vee y] = [x] \cup [y]$.*

*Proof.* (1) Suppose $x \leq y$ and $U \in [x]$. Then $x \in U$, so $y \in U$ ($U$ is closed upward), and $U \in [y]$.

(2) Every ultrafilter on $B$ contains $\top$ and no ultrafilter contains $\bot$.

(3) $\neg x \in U$ if and only if $x \notin U$ ($U$ contains exactly one of $x$ and $\neg x$).

(4) If $U \in [x] \cap [y]$, then $x \in U$ and $y \in U$, so $x \wedge y \in U$ ($U$ is closed under meets), and $U \in [x \wedge y]$. Conversely, if $U \in [x \wedge y]$, then $x \wedge y \in U$, so $x \in U$ and $y \in U$ ($U$ is closed upward), so $U \in [x] \cap [y]$.

(5) Using Exercise 7 and (3) and (4) above,

$$\begin{aligned} [x \vee y] &= [\neg(\neg x \wedge \neg y)] \\ &= S(B) \setminus [\neg x \wedge \neg y] \\ &= S(B) \setminus ([\neg x] \cap [\neg y]) \\ &= (S(B) \setminus [\neg x]) \cup (S(B) \setminus [\neg y]) \\ &= [\neg\neg x] \cup [\neg\neg y] \\ &= [x] \cup [y]. \end{aligned}$$ □

By conditions (2) and (4) of Lemma 2.10, the family $\{[x] \mid x \in B\}$ contains $S(B)$ and is closed under intersection, so it forms a basis for a topology $\tau$ on $S(B)$. Note that each basic open set $[x]$ is in fact clopen, since its complement $[\neg x]$ is also basic open.

A **Stone space** is a compact Hausdorff space with a basis of clopen sets.[10]

**Lemma 2.11.** *The space $S(B)$ is a Stone space.*

*Proof.* We have already observed that $S(B)$ has a basis of clopen sets. For the Hausdorff property, note that if $U \neq V$ are points in $S(B)$, then without loss of generality $U \not\subseteq V$, so there is some $x \in U$ with $x \notin V$. Since $V$ is an ultrafilter, $\neg x \in V$. So $[x]$ and $[\neg x]$ are disjoint open neighborhoods of $U$ and $V$, respectively.

It remains to show compactness. We will prove the complemented form: Suppose $(C_i)_{i \in I}$ is a family of closed sets such that $\bigcap_{i \in I} C_i = \emptyset$. Then already there is a finite subfamily $(C_i)_{i \in J}$, where $J$ is a finite subset of $I$, such that $\bigcap_{i \in J} C_i = \emptyset$. Further, we may assume that each $C_i$ is a basic clopen set, i.e. $[x_i]$ for some $x_i \in B$.

Let $F = \mathrm{Fil}(\{x_i \mid i \in I\})$. Suppose for contradiction that $F$ is proper. Then it extends to an ultrafilter $U$, and since $\{x_i \mid i \in I\} \subseteq F \subseteq U$, we have $U \in \bigcap_{i \in I} [x_i] = \emptyset$, contradiction. Thus $F$ is not proper, and there is some $J \subseteq I$ such that $\bigwedge_{i \in J} x_i = \bot$. Then $[\bigwedge_{i \in J} x_i] = \bigcap_{i \in J} [x_i] = \emptyset$. □

We call $S(B)$ the **Stone space of** $B$. Recall that when $X$ is any topological space, the **clopen algebra of** $X$, $(\mathrm{Cl}(X); \subseteq, X, \emptyset, \cap, \cup, -)$, is the Boolean algebra of clopen sets in $X$.

The Stone duality theorem says that every Boolean algebra is the clopen algebra of some topological space (namely its Stone space), and every Stone space is the Stone space of some Boolean algebra (namely its clopen algebra).

[10]Spaces with a basis of clopen sets are often called *zero-dimensional*; the dimension in question is the *small inductive dimension*. It is a fact that a compact Hausdorff space is zero-dimensional if and only if it is *totally disconnected*, i.e. every connected component is a singleton. We will not prove that here.

Thursday 9/6 **Theorem 2.12** (Stone duality)**.** *For every Boolean algebra $B$, $B \cong \mathrm{Cl}(S(B))$. Conversely, for every Stone space $X$, $X \cong S(\mathrm{Cl}(X))$.*

*Proof.* Let $B$ be a Boolean algebra. Then the map $[-]: B \to \mathrm{Cl}(S(B))$ is defined by $b \mapsto [b]$. Lemma 2.10 exactly says that $[-]$ is a homomorphism. To see that it is an embedding, it suffices to show that if $[x] \subseteq [y]$, then $x \leq y$. Indeed, injectivity follows, since if $[x] = [y]$, then $x \leq y$ and $y \leq x$, so $x = y$.

We show the contrapositive. So assume $x \not\leq y$. By Lemma 2.5, $x \wedge \neg y \neq \bot$. So the filter $\uparrow(x \wedge \neg y)$ is proper and extends to an ultrafilter $U$. Now $x \wedge \neg y \in U$, so $U \in [x]$ and $U \in [\neg y]$, so $U \notin [y]$. It follows that $[x] \not\subseteq [y]$.

It remains to show that $[-]$ is surjective. So let $C \subseteq S(B)$ be a clopen set. Since $C$ is open, we can write it as a union of basic open sets, $C = \bigcup_{i \in I}[x_i]$. Since $C$ is a closed subset of a compact space, it is compact, and the cover $\{[x_i] \mid i \in I\}$ has a finite subcover $\{[x_i] \mid i \in J\}$, where $J$ is a finite subset of $I$. Then $C = \bigcup_{i \in J}[x_i] = [\bigvee_{i \in J} x_i]$.

Conversely, let $X$ be a Stone space. We will use several times the fact that if $p \neq q$ are points of $X$, then there is a clopen set $C$ separating $p$ and $q$, i.e. $p \in C$ and $q \in -C$. Indeed, by Hausdorffness, $p$ has an open neighborhood $V$ such that $q \notin V$. By shrinking $V$, we may assume it is a basic clopen set.

The map $f: X \to S(\mathrm{Cl}(X))$ is given by $p \mapsto U_p = \{C \in \mathrm{Cl}(X) \mid p \in C\}$. There are several things to check:

(a) $U_p$ is an ultrafilter. The clopen sets containing $p$ are closed upward, closed under intersection, include $X$, and do not include $\emptyset$. So $U_p$ is a proper filter. And for any clopen set $C$, either $p \in C$ or $p \in -C$, so $U_x$ is an ultrafilter.

(b) $f$ is injective. If $p \neq q$, then letting $C$ be a clopen set separating $p$ and $q$, we have $C \in U_p$ and $C \notin U_q$, so $U_p \neq U_q$.

(c) $f$ is surjective. Let $U$ be an ultrafilter on $\mathrm{Cl}(X)$. Let $P = \bigcap_{C \in U} C$. We claim that $P$ is a singleton $\{p\}$ and $f(p) = U_p = U$.

   $P$ is nonempty by compactness. Indeed, if $\bigcap_{C \in U} C = \emptyset$, then there are finitely many $C_1, \ldots, C_n \in U$ such that $\bigcap_{i=1}^n C_i = \emptyset$. But $\bigcap_{i=1}^n C_i \in U$, which is a contradiction.

   If $p, q \in P$ and the clopen set $C$ separates $p$ and $q$, then since either $C$ or $-C$ is in $U$, either $p$ or $q$ is not in $P$, contradiction. It follows that $p$ and $q$ are not separated, so $p = q$. Now it is clear that $U \subseteq U_p$, since $p \in C$ for all $c \in U$. And conversely, if $p \in C$, then $-C \notin U$, so $C \in U$, and $U_p \subseteq U$.

(d) $f$ is continuous. A basic clopen set in $S(\mathrm{Cl}(X))$ is of the form $[C]$, where $C$ is a clopen subset of $X$. We claim that $f^{-1}([C]) = C$. Indeed, $f(p) = U_p \in [C]$ iff $C \in U_p$ iff $p \in C$.

(e) $f$ is a homeomorphism. It suffices to show that the image of a closed set is closed. So let $C \subseteq X$ be closed. Then $C$ is compact (a closed subset of a compact space is compact). So $f(C)$ is compact (the continuous image of a compact set is compact). So $f(C)$ is closed (a compact subset of a

Hausdorff space is closed). What we have used here is just the general fact that a continuous bijection from a compact space to a Hausdorff space is a homeomorphism. □

As an immediate corollary, we get the following topology-free representation theorem for Boolean algebras.

**Corollary 2.13.** *Every Boolean algebra $B$ embeds as a subalgebra of a powerset algebra.*

*Proof.* For any topological space $X$, the clopen algebra $\mathrm{Cl}(X)$ is a subalgebra of the powerset algebra $\mathcal{P}(X)$. So the Stone isomorphism $B \cong \mathrm{Cl}(S(B))$ is an embedding $B \hookrightarrow \mathcal{P}(S(B))$. □

**Exercise 10.** (This exercise is not essential; you should only do it if you are already comfortable with the language of category theory.)

1. Let $\mathsf{Bool}$ be the category of Boolean algebras and homomorphisms, and let $\mathsf{Stone}$ be the category of Stone spaces and continuous maps. By defining their action on morphisms, show how to make $S\colon \mathsf{Bool}^{\mathrm{op}} \to \mathsf{Stone}$ and $\mathrm{Cl}\colon \mathsf{Stone}^{\mathrm{op}} \to \mathsf{Bool}$ into contravariant functors between these categories.

2. Show that the pair $(S, \mathrm{Cl})$ forms a contravariant equivalence of categories $\mathsf{Bool} \equiv \mathsf{Stone}^{\mathrm{op}}$. This is the modern formulation of the Stone duality theorem.

3. Let $\mathsf{FinBool}$ and $\mathsf{FinSet}$ be the categories of finite Boolean algebras and finite sets, respectively. Show that $\mathsf{FinBool}$ is equivalent to $\mathsf{FinSet}^{\mathrm{op}}$.

# 3 Proofs and completeness for first-order logic

## 3.1 A proof system

We now return to first-order logic and define the provability relation $\vdash$. Our proof rules are given below.[11]

[11] There are many sound and complete proof systems for first-order logic, i.e. many ways to define the relation $\vdash$. The system presented here is definitely not the most efficient: in terms of the number of rules, or from the point of view of proof theory. But I find it to be well-motivated and well-suited for proving completeness.

**Propositional rules:** In these rules $\varphi$, $\chi$, and $\psi$ are formulas in context $x$.

$$\frac{}{\varphi \vdash_x \varphi}\ \text{R} \qquad \frac{\varphi \vdash_x \chi \quad \chi \vdash_x \psi}{\varphi \vdash_x \psi}\ \text{T}$$

$$\frac{}{\varphi \vdash_x \top}\ \text{TOP} \qquad \frac{}{\chi \wedge \psi \vdash_x \chi}\ \text{AND}_L \qquad \frac{}{\chi \wedge \psi \vdash_x \psi}\ \text{AND}_R \qquad \frac{\varphi \vdash_x \chi \quad \varphi \vdash_x \psi}{\varphi \vdash_x \chi \wedge \psi}\ \text{AND}$$

$$\frac{}{\bot \vdash_x \varphi}\ \text{BOT} \qquad \frac{}{\chi \vdash_x \chi \vee \psi}\ \text{OR}_L \qquad \frac{}{\psi \vdash_x \chi \vee \psi}\ \text{OR}_R \qquad \frac{\chi \vdash_x \varphi \quad \psi \vdash_x \varphi}{\chi \vee \psi \vdash_x \varphi}\ \text{OR}$$

$$\frac{}{\varphi \wedge (\chi \vee \psi) \vdash_x (\varphi \wedge \chi) \vee (\varphi \wedge \psi)}\ \text{D} \qquad \frac{}{\varphi \wedge \neg\varphi \vdash_x \bot}\ \text{NOT}_1 \qquad \frac{}{\top \vdash_x \varphi \vee \neg\varphi}\ \text{NOT}_2$$

**Equality rules:** In these rules, $t$, $t'$, and $t''$ are terms of type $s$ in context $x$ for some $s \in \mathcal{S}$. Also, $t_i$ is a term of type $s_i$ in context $x$ for all $1 \leq i \leq n$, $f$ is a function symbol of type $(s_1, \ldots, s_n) \to s$, and $R$ is a relation symbol of type $(s_1, \ldots, s_n)$.

$$\frac{}{\top \vdash_x t = t}\ \text{R}_= \qquad \frac{}{t = t' \vdash_x t' = t}\ \text{S}_= \qquad \frac{}{t = t' \wedge t' = t'' \vdash_x t = t''}\ \text{T}_=$$

$$\frac{}{\bigwedge_{i=1}^n t_i = t_i' \vdash_x f(t_1, \ldots, t_n) = f(t_1', \ldots, t_n')}\ \text{FUN}$$

$$\frac{}{(\bigwedge_{i=1}^n t_i = t_i') \wedge R(t_1, \ldots, t_n) \vdash_x R(t_1', \ldots, t_n')}\ \text{REL}$$

**Quantifier rules:** In these rules, $y$ is a single variable of type $s$ not in context $x$, $t$ is a term of type $s$ in context $x$, $\varphi$ is a formula in context $xy$, and $\psi$ is a formula in context $x$ in which $y$ is not bound (so we an also view it as a formula in context $xy$).

$$\frac{}{\varphi[y \mapsto t] \vdash_x \exists y\, \varphi}\ \text{SUB}(t) \qquad \frac{\varphi \vdash_{xy} \psi}{\exists y\, \varphi \vdash_x \psi}\ \text{E}$$

In the rule ($\text{SUB}(t)$), the notation $\varphi[y \mapsto t]$ means that the term $t$ is substituted for every instance of the variable $y$ appearing in $\varphi$. Note that this substitution is only valid when $y$ and $t$ have the same type. If $\varphi$ is a formula in context $xy$ and $t$ is a term in context $x$, then we view $\varphi[y \mapsto t]$ as a formula in context $x$. When we make the context explicit by writing the formula as $\varphi(x, y)$, we also write the new formula as $\varphi(x, t)$.

The expression $\varphi \vdash_x \psi$ is called a **sequent**. Notice that every sequent is decorated by a variable context $x$, and that both formulas $\varphi$ and $\psi$ are in context $x$. We just write $\vdash$ instead of $\vdash_{()}$ when the the context is empty.

In a rule, the sequents appearing above the horizontal line are the **premises**, and the sequent appearing below the horizontal line is the **conclusion**. A **proof tree** is a finite tree, with each node labeled by a sequent. A proof tree is **valid** if for every node, there is some rule such that the node is labeled by the conclusion, and the children of the node are labeled by the premises, of an instance of that rule. We assert $\varphi \vdash \psi$ if this sequent labels the root of a valid proof tree.

**Example 3.1.** Let's say our goal is to prove $P \vdash \neg\neg P$, when $P$ is a propositional symbol. The following is a valid proof tree:

$$\dfrac{\dfrac{\dfrac{}{P \vdash P}\,\textsc{r} \quad \dfrac{\dfrac{}{P \vdash \top}\,\textsc{top} \quad \dfrac{}{\top \vdash \neg P \vee \neg\neg P}\,\textsc{not}_2}{P \vdash \neg P \vee \neg\neg P}\,\textsc{t}}{P \vdash P \wedge (\neg P \vee \neg\neg P)}\,\textsc{and} \qquad \dfrac{}{P \wedge (\neg P \vee \neg\neg P) \vdash (P \wedge \neg P) \vee (P \wedge \neg\neg P)}\,\textsc{d}}{P \vdash (P \wedge \neg P) \vee (P \wedge \neg\neg P)}\,\textsc{t}$$

Here is another one:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{P \wedge \neg P \vdash \bot}\,\textsc{not}_1 \quad \dfrac{}{\bot \vdash P \wedge \neg\neg P}\,\textsc{bot}}{P \wedge \neg P \vdash P \wedge \neg\neg P}\,\textsc{t} \qquad \dfrac{}{P \wedge \neg\neg P \vdash P \wedge \neg\neg P}\,\textsc{r}}{(P \wedge \neg P) \vee (P \wedge \neg\neg P) \vdash P \wedge \neg\neg P}\,\textsc{or} \qquad \dfrac{}{P \wedge \neg\neg P \vdash \neg\neg P}\,\textsc{and}_R}{(P \wedge \neg P) \vee (P \wedge \neg\neg P) \vdash \neg\neg P}\,\textsc{t}$$

Putting these together yields $P \vdash \neg\neg P$:

$$\dfrac{\begin{matrix}\vdots \\ P \vdash (P \wedge \neg P) \vee (P \wedge \neg\neg P)\end{matrix} \quad \begin{matrix}\vdots \\ (P \wedge \neg P) \vee (P \wedge \neg\neg P) \vdash \neg\neg P\end{matrix}}{P \vdash \neg\neg P}\,\textsc{t}$$

**Example 3.2.** Here is an example of tricky things you can do by shuffling variables. Let $x$ and $y$ be single variables of the same type, let $\varphi(y)$ be a formula in context $y$, and let $\varphi'(x)$ be $\varphi[y \mapsto x]$, obtained by substituting $x$ for $y$ everywhere. The following proof tree shows that $\exists x\, \varphi' \vdash \exists y\, \varphi$.

$$\dfrac{\dfrac{}{\varphi' \vdash_x \exists y\, \varphi}\,\textsc{sub}(x)}{\exists x\, \varphi' \vdash \exists y\, \varphi}\,\textsc{e}$$

In the first line, we view $\exists y\, \varphi$ in the context $x$ (although $x$ is not mentioned in this sentence). Since $\varphi'$ is $\varphi[y \mapsto x]$, this is a valid instance of ($\textsc{sub}(x)$). Then we can use ($\textsc{e}$) to introduce the quantifier and remove $x$ from the context, since $x$ is not mentioned in $\exists y\, \varphi$.

We will make use of the fact that $\exists x\, \varphi' \vdash \exists y\, \varphi$ in the proof of completeness.

**Remark 3.3.** The proof rules probably look familiar to you: they are essentially our axioms for Boolean algebras. Let's make that precise.

Fix a variable context $x$, and recall that $\mathcal{L}_x$ is the set of all formulas in context $x$. By ($\textsc{r}$) and ($\textsc{t}$), the relation $\vdash$ is a preorder on $\mathcal{L}_x$. We pass to the associated partial order $\widehat{\mathcal{L}_x}$, writing $\widehat{\varphi}$ for the equivalence class of a sentence $\varphi$

(so $\widehat{\varphi} = \widehat{\psi}$ if and only if $\varphi \vdash \psi$ and $\psi \vdash \varphi$). We say that two sentences in the same equivalence class are **logically equivalent**.

Now by (TOP) and (BOT), this partial order has a top element $\widehat{\top}$ and a bottom element $\widehat{\bot}$. By ($\text{AND}_L$), ($\text{AND}_R$), and (AND), $\widehat{\chi \wedge \psi}$ is the greatest lower bound of $\widehat{\chi}$ and $\widehat{\psi}$, and by ($\text{OR}_L$), ($\text{OR}_R$), and (OR), $\widehat{\chi \vee \psi}$ is the least upper bound of $\widehat{\chi}$ and $\widehat{\psi}$. So the partial order is a lattice, with meet and join as above.

By (D), the lattice is distributive. There is something to be checked here, since (D) only gives one inequality of the distributive law, namely

$$x \wedge (y \vee z) \leq (x \wedge y) \vee (x \wedge z).$$

But the other inequality holds in any lattice. Indeed, note that $x \wedge y \leq x$ and $x \wedge z \leq x$, so

$$(x \wedge y) \vee (x \wedge z) \leq x.$$

Similarly, $x \wedge y \leq y \leq y \vee z$, and $x \wedge z \leq z \leq y \vee z$, so

$$(x \wedge y) \vee (x \wedge z) \leq y \vee z.$$

It follows that

$$(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee z).$$

Finally, by ($\text{NOT}_1$) and ($\text{NOT}_2$), every element $\widehat{\varphi}$ has a complement $\widehat{\neg\varphi}$. So $\widehat{\mathcal{L}_x}$ is a Boolean algebra.

From this observation, we can already dispense with the hassle of constructing explicit proof trees in many situations: for example, since the $\vdash$ relation is the order relation on the Boolean algebra of sentences, syntactic $\neg$ is the complement operation in this Boolean algebra, and $x = \neg\neg x$ in any Boolean algebra, we know that $\varphi \vdash_x \neg\neg\varphi$, and also $\neg\neg\varphi \vdash_x \varphi$, for any formula $\varphi$ in context $x$, without having to write out proof trees.

In the future, we will often dispense with the $\widehat{\varphi}$ notation, identifying a formula with its logical equivalence class when there is no danger in doing so.

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When $T$ is a theory and $\varphi$ is a sentence, we write $T \vdash \varphi$, read $T$ **proves** $\varphi$, if there are finitely many sentences $\psi_1, \ldots \psi_n \in T$ such that $\bigwedge_{i=1}^n \psi_i \vdash \varphi$. We say that $T$ is **inconsistent** if $T \vdash \bot$, and otherwise $T$ is **consistent**.

Note for any theory $T$, the set $F_T = \{\varphi \in \mathcal{L}_{()} \mid T \vdash \varphi\}$ of sentences provable from $T$ is exactly the filter generated by $T$ in the Boolean algebra of sentences. This filter is proper if and only if $T$ is consistent. And this filter is an ultrafilter if and only if $T$ is **complete (for provability)**, i.e. for every sentence $\varphi$, $T \vdash \varphi$ or $T \vdash \neg\varphi$. In the remainder of this section when we say $T$ is complete, we mean complete for provability. It will be a consequence of soundness and completeness that a theory is complete for provability if and only if it is complete (as defined in Section 1.6 in terms of $\models$).

**Lemma 3.4.** *Suppose $T$ is a consistent theory. Then there is a complete theory $T^*$ such that $T \subseteq T^*$.*

*Proof.* Since $T$ is consistent, the associated filter $F_T = \{\varphi \in \mathcal{L}_{()} \mid T \vdash \varphi\}$ is proper. By the ultrafilter lemma, $F_T$ extends to an ultrafilter on $\mathcal{L}_{()}$, i.e. a complete theory $T^*$, and $T \subseteq F_T \subseteq T^*$. □

## 3.2 Soundness and completeness

We now prove that the syntactic relation $\vdash$ is equal to the semantic relation $\models$.

**Theorem 3.5** (Soundness)**.** *If $T \vdash \varphi$, then $T \models \varphi$.*

*Proof.* We say a sequent $\varphi \vdash_x \psi$ is **semantically valid** if for every structure $A$ and every interpretation $a$ of the context $x$, if $A \models \varphi(a)$, then $A \models \psi(a)$.

**Claim:** Every sequent which labels the root of a valid proof tree is semantically valid.

The argument is by induction on the complexity of the proof tree. In the base case, we must show that for every rule with no premises, every instance of that rule is semantically valid. In the inductive step, for every instance of every rule with premises, we may assume that the premises are semantically valid, and we must show that the conclusion is semantically valid.

There are many cases to check, but they are all straightforward. So we just illustrate a few for example. Consider the rule (AND):

$$\frac{\varphi \vdash_x \chi \quad \varphi \vdash_x \psi}{\varphi \vdash_x \chi \wedge \psi}\ \text{AND}.$$

We may assume by induction that the premises are semantically valid. Let $A$ be a structure and $a \in A^x$ such that $A \models \varphi(a)$. By the induction hypothesis, $A \models \chi(a)$ and $A \models \psi(a)$, so $A \models (\chi \wedge \psi)(a)$.

Consider the rule (REL):

$$\frac{}{(\bigwedge_{i=1}^n t_i = t_i') \wedge R(t_1, \dots, t_n) \vdash_x R(t_1', \dots, t_n')}\ \text{REL}$$

This is a base case. Let $A$ be a structure and $a \in A^x$ such that

$$A \models ((\bigwedge_{i=1}^n t_i = t_i') \wedge R(t_1, \dots, t_n))(a).$$

Then $(t_1^A(a), \dots, t_n^A(a)) \in R^A$. But also $t_i^A(a) = (t_i')^A(a)$ for all $1 \leq i \leq n$, so also $((t_1')^A(a), \dots, (t_n')^A(a)) \in R^A$, and $A \models R(t_1', \dots, t_n')(a)$.

The quantifier rules are the most complicated, so we illustrate both of them.

$$\frac{}{\varphi[y \mapsto t] \vdash_x \exists y\, \varphi}\ \text{SUB}(t)$$

Let $A$ be a structure and $a \in A^x$ such that $A \models \varphi[y \mapsto t](a)$. That is, writing $\varphi$ as $\varphi(x, y)$, we have $A \models \varphi(a, t^A(a))$. So $A \models \exists y\, \varphi(a, y)$, witnessed by the element $t^A(a)$.

$$\frac{\varphi \vdash_{xy} \psi}{\exists y\, \varphi \vdash_x \psi}\ \text{E}$$

We may assume by induction that $\varphi \vdash_{xy} \psi$ is semantically valid. Let $A$ be a structure and $a \in A^x$ such that $A \models \exists y\, \varphi(a, y)$. Then there is some $b \in A_s$

such that $A \models \varphi(a, b)$. By the induction hypothesis, $A \models \psi(a, b)$, but $\psi(x)$ is a formula in context $x$ (it does not mention the variable $y$), so $A \models \psi(a)$.

Having established the claim, assume $T \vdash \varphi$. Then there is a finite subset $\Sigma \subseteq_{\text{fin}} T$ such that $\bigwedge_{\psi \in \Sigma} \psi \vdash \varphi$. By the claim, every model of $\bigwedge_{\psi \in \Sigma} \psi$ satisfies $\varphi$. And since every model of $T$ is a model of $\bigwedge_{\psi \in \Sigma} \psi$, we have $T \models \varphi$. $\square$

So soundness just amounts to checking that the proof rules don't say anything wrong. We now embark on proving the converse, completeness, which is much more involved.

We first prove an a priori weaker claim: Any consistent theory has a model. In other words, if $T \nvdash \perp$, then $T \not\models \perp$. How can we take a syntactic assumption, the consistency of $T$, and produce an actual model of $T$? We take a cue from the theory of free algebraic structures (e.g. free groups or the "term algebra" from Exercise 2) and construct the model from the syntax of $T$.

**Construction 3.6.** Let $T$ be any $\mathcal{V}$-theory. For each sort $s \in \mathcal{S}$, recall that $\mathcal{T}_s$ is the set of $\mathcal{V}$-terms of type $s$ in the empty context. Define a relation $\sim_T$ on $\mathcal{T}_s$ by $t \sim_T t'$ if and only if $T \vdash (t = t')$.

By ($\text{R}_=$), ($\text{S}_=$), and ($\text{T}_=$), $\sim_T$ is an equivalence relation on $\mathcal{T}_s$. We denote by $[t]$ the equivalence class of $t$.

Let $M(T)_s = \mathcal{T}_s / {\sim_T}$. We will define a $\mathcal{V}$-structure $M(T)$ with domain $(M(T)_s)_{s \in \mathcal{S}}$.

For each function symbol $f$ of type $(s_1, \ldots, s_n) \to s$, define

$$f^M([t_1], \ldots, [t_n]) = [f(t_1, \ldots, t_n)].$$

This is well-defined: if $t_i \sim_T t'_i$ for $1 \leq i \leq n$, then $T \models \bigwedge_{i=1}^n t_i = t'_i$, so by (FUN), $T \models f(t_1, \ldots, t_n) = f(t'_1, \ldots, t'_n)$, and $f(t_1, \ldots, t_n) \sim_T f(t'_1, \ldots, t'_n)$.

For each relation symbol $R$ of type $(S_1, \ldots, S_n)$, define

$$([t_1], \ldots, [t_n]) \in R^M \iff T \vdash R(t_1, \ldots, t_n).$$

This is well-defined: if $t_i \sim_T t'_i$ for $1 \leq i \leq n$ and $T \vdash R(t_1, \ldots, t_n)$, then $T \models (\bigwedge_{i=1}^n t_i = t'_i) \wedge R(t_1, \ldots, t_n)$, so by (REL), $T \models R(t'_1, \ldots, t'_n)$.

Our equality rules ensure that the $\mathcal{V}$-structure $M(T)$ is well-defined, but we'd really like it to be a model of $T$. The problem is that $T$ might contain a sentence with a quantifier, like $\exists x\, \varphi(x)$, and there is no guarantee that there is a term $t$ in the empty context such that $M(T) \models \varphi(t)$. In fact, if $\mathcal{V}$ has no constant symbols, then there are no $\mathcal{V}$-terms in the empty context at all, and $M(T)$ is empty! The following definition rectifies this situation.

A theory $T$ has **Henkin witnesses** if for any formula $\varphi(y)$ in a context with a single variable $y$ of type $s$, if $T \vdash \exists y\, \varphi(y)$, then there is a constant symbol $c_\varphi$ of type $s$ such that $T \vdash \varphi(c_\varphi)$.

**Lemma 3.7.** *Suppose $T$ is a complete consistent theory with Henkin witnesses. Then $M(T) \models T$.*

*Proof.* The proof amounts to understanding the interpretations of terms and formulas in $M(T)$. We proceed by induction on the structure of terms and formulas. This proof has a feature common to all such inductions: while we only care about whether $M(T) \models \varphi$ when $\varphi$ is a *sentence*, in order to carry out the induction, we have to formulate our claims for formulas in an arbitrary variable context.

**Claim 1:** If $t(x)$ is a term in context $x = (x_1, \ldots, x_n)$ and $([t_1], \ldots, [t_n])$ is an interpretation of $x$ in $M(T)$, then $t^{M(T)}([t_1], \ldots, [t_n]) = [t(t_1, \ldots, t_n)]$. Here $t(t_1, \ldots, t_n)$ is the term in the empty context obtained by substituting $t_i$ for $x_i$ in $t$ for all $1 \leq i \leq n$.

The proof is by induction on the complexity of terms.

- If $t$ is a variable $x_i$, then $t^{M(T)}([t_1], \ldots, [t_n]) = [t_i] = [t(t_1, \ldots, t_n)]$.
- If $t$ is $f(u_1, \ldots, u_m)$, where $u_1, \ldots, u_m$ are terms in context $x$, then:

$$\begin{aligned} t^{M(T)}([t_1], \ldots, [t_n]) &= f^{M(T)}(u_1^{M(T)}([t_1], \ldots, [t_n]), \ldots, u_m^{M(T)}([t_1], \ldots, [t_n])) \\ &= f^{M(T)}([u_1(t_1, \ldots, t_n)], \ldots, [u_m(t_1, \ldots, t_n)]) \\ &= [f(u_1(t_1, \ldots, t_n), \ldots, u_m(t_1, \ldots, t_n))] \\ &= [t(t_1, \ldots, t_n)]. \end{aligned}$$

**Claim 2:** If $\varphi(x)$ is a formula in context $x = (x_1, \ldots, x_n)$ and $([t_1], \ldots, [t_n])$ is an interpretation of $x$ in $M(T)$, then $M(T) \models \varphi([t_1], \ldots, [t_n])$ if and only if $T \vdash \varphi(t_1, \ldots, t_n)$. Here $\varphi(t_1, \ldots, t_n)$ is the sentence obtained by substituting $t_i$ for $x_i$ in $\varphi$ for all $1 \leq i \leq n$.

The proof is by induction on the complexity of formulas.

- If $\varphi$ is $t = t'$, where $t$ and $t'$ are terms of type $s \in \mathcal{S}$ in context $x$, then

$$\begin{aligned} M(T) \models \varphi([t_1], \ldots, [t_n]) \quad &\text{iff} \quad t^{M(T)}([t_1], \ldots, [t_n]) = t'^{M(T)}([t_1], \ldots, [t_n]) \\ &\text{iff} \quad [t(t_1, \ldots, t_n)] \sim_T [t'(t_1, \ldots, t_n)] \\ &\text{iff} \quad T \vdash t(t_1, \ldots, t_n) = t'(t_1, \ldots, t_n) \\ &\text{iff} \quad T \vdash \varphi(t_1, \ldots, t_n). \end{aligned}$$

- If $\varphi$ is $R(u_1, \ldots, u_m)$, where $u_1, \ldots, u_m$ are terms in context $x$, then

$$\begin{aligned} M(T) \models \varphi([t_1], \ldots, [t_n]) \quad &\text{iff} \quad (u_i^{M(T)}([t_1], \ldots, [t_n]))_{i=1}^m \in R^{M(T)} \\ &\text{iff} \quad ([u_i(t_1, \ldots, t_n)])_{i=1}^m \in R^{M(T)} \\ &\text{iff} \quad T \vdash R(u_1(t_1, \ldots, t_n), \ldots, u_m(t_1, \ldots, t_n)) \\ &\text{iff} \quad T \vdash \varphi(t_1, \ldots, t_n). \end{aligned}$$

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- If $\varphi$ is $\top$, then $M(T) \models \varphi([t_1], \ldots, [t_n])$ and $T \vdash \varphi(t_1, \ldots, t_n)$.

- If $\varphi$ is $\bot$, then $M(T) \not\models \varphi([t_1], \ldots, [t_n])$ and $T \nvdash \varphi(t_1, \ldots, t_n)$, since $T$ is consistent.

- If $\varphi$ is $\chi \wedge \psi$, then

$$\begin{aligned} M(T) \models \varphi([t_1], \ldots, [t_n]) \quad &\text{iff} \quad M(T) \models \chi([t_1], \ldots, [t_n]) \\ &\qquad \text{and } M(T) \models \psi([t_1], \ldots, [t_n]) \\ &\text{iff} \quad T \vdash \chi(t_1, \ldots, t_n) \text{ and } T \vdash \psi(t_1, \ldots, t_n) \\ &\text{iff} \quad T \vdash \varphi(t_1, \ldots, t_n). \end{aligned}$$

- If $\varphi$ is $\chi \vee \psi$, then

$$\begin{aligned} M(T) \models \varphi([t_1], \ldots, [t_n]) \quad &\text{iff} \quad M(T) \models \chi([t_1], \ldots, [t_n]) \\ &\qquad \text{or } M(T) \models \psi([t_1], \ldots, [t_n]) \\ &\text{iff} \quad T \vdash \chi(t_1, \ldots, t_n) \text{ or } T \vdash \psi(t_1, \ldots, t_n) \\ &\text{iff} \quad T \vdash \varphi(t_1, \ldots, t_n). \end{aligned}$$

  Here we used completeness: recall (from Lemma 2.10) that an ultrafilter contains a join $x \vee y$ if and only if it contains $x$ or it contains $y$.

- If $\varphi$ is $\neg\chi$, then

$$\begin{aligned} M(T) \models \varphi([t_1], \ldots, [t_n]) \quad &\text{iff} \quad M(T) \not\models \chi([t_1], \ldots, [t_n]) \\ &\text{iff} \quad T \nvdash \chi(t_1, \ldots, t_n) \\ &\text{iff} \quad T \vdash \varphi(t_1, \ldots, t_n), \end{aligned}$$

  since $T$ is complete.

- If $\varphi$ is $\exists y\, \psi(x, y)$, suppose $M(T) \models \varphi([t_1], \ldots, [t_n])$. Then there is some $[u] \in M(T)^y$ such that $M(T) \models \psi([t_1], \ldots, [t_n], [u])$. By induction, $T \vdash \psi(t_1, \ldots, t_n, u)$. By ($\textsc{sub}(u)$), $T \vdash \exists y\, \psi(t_1, \ldots, t_n, y)$, so $T \vdash \varphi(t_1, \ldots, t_n)$.

  Conversely, suppose $T \vdash \varphi(t_1, \ldots, t_n)$. So $T \vdash \exists y\, \psi(t_1, \ldots, t_n, y)$. Let $\theta(y)$ be $\psi(t_1, \ldots, t_n, y)$. Since $T$ has Henkin witnesses, $T \vdash \psi(t_1, \ldots, t_n, c_\theta)$. By induction, $M(T) \models \psi([t_1], \ldots, [t_n], [c_\theta])$, so $M(T) \models \varphi([t_1], \ldots, [t_n])$.

Now for every sentence $\varphi \in T$, we have $T \vdash \varphi$, so $M(T) \models \varphi$ by Claim 2. So $M(T) \models T$, as was to be shown. □

Of course, not every theory has Henkin witnesses. So our next goal is to take an arbitrary $\mathcal{V}$-theory $T$ and extend it to a complete $\mathcal{V}'$-theory $T'$ with Henkin witnesses, where $\mathcal{V}'$ is a vocabulary obtained by adding new constant symbols to $\mathcal{V}$. It is in this step where we need to do some real proof-theoretic work, so we pause to make some observations about our proof system.

(1) Which instances of the rule ($\textsc{sub}(t)$) are available to us depends on the vocabulary. For example, if $y$ is a variable of type $s$, the sentence $\exists y\, \top$ is

true in a structure if and only if the sort $s$ is nonempty. If $\mathcal{V}$ has a term $t$ of type $s$ in the empty context (for example, a constant symbol of type $s$), then we have $\top \vdash \exists x\, \top$ by $(\textsc{sub}(t))$, since $\top[x \mapsto t] = \top$.

The sequent $\top \vdash \exists x\, \top$ is semantically valid, since sort $s$ is nonempty in every $\mathcal{V}$-structure (it contains at least the interpretation of the term $t$). On the other hand, if $\mathcal{V}$ has no terms of sort $s$ in the empty context, then there are $\mathcal{V}$-structures in which the sort $s$ is empty, and $\top \nvdash \exists x\, \top$.

Note that the term $t$ does not actually appear in the proof tree for $\top \vdash \exists x\, \top$, just in the rule! This is why we make $t$ explicit when writing $(\textsc{sub}(t))$. When we say that a symbol or a variable occurs in a proof tree, we mean that it appears in any of the sentences in the sequents in the tree, *or* in a term $t$ used in an instance of $(\textsc{sub}(t))$. When there could be any confusion about which vocabulary a proof takes place in, we will be careful to specify it.

(2) If $\varphi \vdash_x \psi$ in vocabulary $\mathcal{V}$, and $\mathcal{V} \subseteq \mathcal{V}'$, then also $\varphi \vdash_x \psi$ in vocabulary $\mathcal{V}'$. In the other direction, a proof in vocabulary $\mathcal{V}$ is a finite tree, and each instance of a rule uses only finitely many symbols in $\mathcal{V}$, so if $\varphi \vdash_x \psi$, then there is a finite vocabulary $\mathcal{V}' \subseteq \mathcal{V}$ (finitely many sorts, function symbols, and relation symbols) such that $\varphi \vdash_x \psi$ in vocabulary $\mathcal{V}'$.

(3) Given a valid proof tree, a variable $y$ which occurs in the tree, and a variable $y'$ of the same type which does not occur in the tree, we can replace $y$ by $y'$ everywhere in the tree, and the resulting tree is still valid. This can be checked by examining each rule and noting that replacing $y$ by $y'$ results in an instance of the same rule.

(4) By Exercise 7, if $\varphi \vdash_x \psi$, then $\neg\psi \vdash_x \neg\varphi$. We will use this contrapositive trick in applications of the rule $(\textsc{e})$. Specifically, suppose $\varphi$ is a formula in context $xy$ and $\psi$ is a formula in context $x$ in which $y$ is not bound. If $\psi \vdash_{xy} \neg\varphi$, then $\psi \vdash_x \neg\exists y\, \varphi$.

Proof: If $\psi \vdash_{xy} \neg\varphi$, then $\varphi \vdash_{xy} \neg\psi$. By $(\textsc{e})$, $\exists y\, \varphi \vdash_x \neg\psi$, so $\psi \vdash_x \neg\exists y\, \varphi$.

**Lemma 3.8** (Lifting constants)**.** *Suppose $\mathcal{V}' = \mathcal{V} \cup \{c\}$, where $c$ is a constant symbol of type $s$. Let $\varphi$ and $\psi$ be $\mathcal{V}'$-formulas in context $x$ such that $\varphi \vdash_x \psi$. Let $z$ be a variable of type $s$ which is not in the context $x$ and does not appear bound in $\varphi$ or $\psi$. Then $\varphi[c \mapsto z] \vdash_{xz} \psi[c \mapsto z]$ via a proof in the vocabulary $\mathcal{V}$.*

*Proof.* By point (3) above, we may assume that $z$ does not occur in the proof tree for $\varphi \vdash_x \psi$. Then we can check, rule by rule, that if we replace the constant symbol $c$ by the variable $z$ (which does not occur in the rule) and add $z$ to the contexts of all terms and formulas in the premises and conclusions, we get another instance of the same rule in vocabulary $\mathcal{V}$.

The least trivial case is the rule $(\textsc{sub}(t))$. So suppose we have an instance of this rule:

$$\frac{}{\varphi[y \mapsto t] \vdash_x \exists y\, \varphi}\ \textsc{sub}(t)$$

where $\varphi$ is a $\mathcal{V}'$-formula in context $xy$, and $t$ is a $\mathcal{V}'$-term in context $x$. Let $t' = t[c \mapsto z]$, and let $\varphi' = \varphi[c \mapsto z]$. We view $t'$ as a $\mathcal{V}$-term in context $xz$ and $\varphi'$ as a $\mathcal{V}$-formula in context $xzy$. Then

$$\frac{}{\varphi'[y \mapsto t'] \vdash_{xz} \exists y\, \varphi'} \ \textsc{sub}(t')$$

is an instance of $(\textsc{sub}(t'))$ in vocabulary $\mathcal{V}$, and $\exists y\, \varphi' = (\exists y\, \varphi)[c \mapsto z]$ and $\varphi'[y \mapsto t'] = (\varphi[y \mapsto t])[c \mapsto z]$, so the conclusion is the sequent obtained by replacing $c$ by $z$ everywhere in the previous instance. $\square$

The lemma on lifting constants is the one place in the proof of completeness where we use the fact that all rules are stated for formulas in arbitrary variable contexts, not just for sentences.

**Lemma 3.9.** *Suppose $T$ is a consistent $\mathcal{V}$-theory. Then there is a vocabulary $\mathcal{V}'$ extending $\mathcal{V}$ by new constant symbols and a complete consistent $\mathcal{V}'$-theory $T'$ with Henkin witnesses such that $T \subseteq T'$.*

*Proof.* The idea is simple: We just add the necessary constant symbols to the vocabulary and the necessary sentences to the theory. After showing that we don't lose consistency, we take a completion. But now that we have added new constant symbols to the theory, there are new formulas that need Henkin witnesses themselves. So we have to repeat the process infinitely many times.

Let $\mathcal{V}_0 = \mathcal{V}$. Let $T = T_0$. By Lemma 3.4, there is a complete consistent $\mathcal{V}_0$-theory $T_0'$ with $T = T_0 \subseteq T_0'$.

Suppose we have constructed a vocabulary $\mathcal{V}_i$ and a complete consistent $\mathcal{V}_i$-theory $T_i'$. For each $\mathcal{V}_i$-formula $\varphi(y)$ in a context with a single variable $y$ of sort $s$, such that $T_i' \vdash \exists y\, \varphi(y)$, let $c_\varphi$ be a new constant symbol of sort $s$. Let $\mathcal{V}_{i+1}$ be the vocabulary obtained by adding all such constant symbols to $\mathcal{V}_i$. And let $T_{i+1} = T_i' \cup \{\varphi(c_\varphi) \mid c_\varphi \in \mathcal{V}_{i+1} \setminus \mathcal{V}_i\}$.

**Claim 1:** If we view $T_i'$ as a $\mathcal{V}_{i+1}$-theory, it is consistent.

It suffices to show that for any finite set $C = \{c_{\varphi_1}, \ldots, c_{\varphi_m}\}$ of constant symbols in $\mathcal{V}_{i+1} \setminus \mathcal{V}_i$, $T_i'$ is consistent in vocabulary $\mathcal{V}_i(C) = \mathcal{V}_i \cup C$. Indeed, if $T_i' \vdash \bot$ in vocabulary $\mathcal{V}_{i+1}$, by observation (2) above the proof only uses finitely many of the new constant symbols. We proceed by induction on $m = |C|$.

In the base case, when $m = 0$, we know that $T_i'$ is a consistent $\mathcal{V}_i$-theory. So suppose $m \geq 1$ and $C = \{c_{\varphi_1}, \ldots, c_{\varphi_m}\}$ is a set of $m$ new constant symbols. Let $C^- = \{c_{\varphi_1}, \ldots, c_{\varphi_{m-1}}\}$. By induction, $T_i'$ is a consistent $\mathcal{V}_i(C^-)$-theory. If $T_i' \vdash \bot$ in vocabulary $\mathcal{V}_i(C)$, then there is sentence $\psi$, which is a conjunction of finitely many sentences in $T_i'$, such that $\psi \vdash \bot$ in $\mathcal{V}_i(C)$. Let $c_{\varphi_m}$ have type $s$. By the lemma on lifting constants, we have $\psi[c_{\varphi_m} \mapsto z] \vdash_z \bot$ in $\mathcal{V}_i(C^-)$, where $z$ is a variable of type $s$ which is not bound in $\psi$. Since $c_{\varphi_m}$ does not appear in $\psi$, $\psi[c_{\varphi_m} \mapsto z]$ is just $\psi(z)$, the sentence $\psi$ viewed in context $z$.

Let $\varphi_m' = \varphi_m[y \mapsto z]$. By ($\textsc{bot}$), $\psi \vdash_z \neg\varphi_m'$, and by observation (4) above (using ($\textsc{e}$)), $\psi \vdash \neg\exists z\, \varphi_m'$, and $T_i' \vdash \neg\exists z\, \varphi_m'$ in $\mathcal{V}(C^-)$.

But also $T_i' \vdash \exists y\, \varphi_m$ in $\mathcal{V}_i$, and hence in $\mathcal{V}_i(C^-)$ by observation (2), so $T_i' \vdash \exists z\, \varphi_m'$ in $\mathcal{V}_i(C^-)$ by Example 3.2. We have shown that $T_i'$ is an inconsistent $\mathcal{V}_i(C^-)$-theory, contradicting the inductive hypothesis.

**Claim 2:** $T_{i+1}$ is consistent.

All proofs in the proof of this claim take place in $\mathcal{V}_{i+1}$. It suffices to show that for any finite set $S = \{\varphi_1(c_{\varphi_1}), \ldots, \varphi_m(c_{\varphi_m})\}$ of sentences in $T_{i+1} \setminus T_i'$, the theory $T_i' \cup S$ is consistent. Indeed, if $T_{i+1} \vdash \bot$, this is witnessed by $\psi \vdash \bot$, where $\psi$ is a finite conjunction of sentences from $T_{i+1}$. We proceed by induction on $m = |S|$.

In the base case, when $m = 0$, we know that $T_i'$ is a consistent $\mathcal{V}_{i+1}$-theory by Claim 1. So suppose $m \geq 1$, and $S = \{\varphi_1(c_{\varphi_1}), \ldots, \varphi_m(c_{\varphi_m})\}$. Let $S^- = \{\varphi_1(c_{\varphi_1}), \ldots, \varphi_{m-1}(c_{\varphi_{m-1}})\}$. By induction, $T_i' \cup S^-$ is consistent. If $T_i' \cup S \vdash \bot$, then there is a finite conjunction $\psi$ of sentences from $T_i' \cup S^-$ such that $\psi \wedge \varphi_m(c_{\varphi_m}) \vdash \bot$.

By Lemma 2.5, $\psi \vdash \neg\varphi_m(c_{\varphi_m})$. By the lemma on lifting constants, we have $\psi[c_{\varphi_m} \mapsto z] \vdash_z \neg\varphi_m(c_{\varphi_m})[c_{\varphi_m} \mapsto z]$, where $z$ is a variable which does not appear bound in $\psi$ or $\varphi_m$. Since $c_{\varphi_m}$ does not occur in $\psi$, $\psi[c_{\varphi_m} \mapsto z]$ is just $\psi(z)$, the sentence $\psi$ viewed in context $z$. Since also $c_{\varphi_m}$ does not occur in the original $\mathcal{V}_i$-formula $\varphi_m(y)$, we have $\varphi_m(c_{\varphi_m})[c_{\varphi_m} \mapsto z] = \varphi_m[y \mapsto z]$. Call this formula $\varphi_m'$.

Then we have $\psi \vdash_z \neg\varphi_m'$. By observation (4) above (using (E)), $\psi \vdash \neg\exists z\, \varphi_m'$, and $T_i' \cup S^- \vdash \neg\exists z\, \varphi_m'$.

But just as in Claim 1, we also have $T_i' \vdash \exists y\, \varphi_m$ in $\mathcal{V}_i$, and hence $T_i' \cup S^- \vdash \exists z\, \varphi_m'$ in $\mathcal{V}_{i+1}$ by Example 3.2 and observation (2). We have shown that $T_i' \cup S^-$ is inconsistent, contradicting the inductive hypothesis.

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We finish the inductive step by defining $T_{i+1}'$ to be a consistent completion of $T_{i+1}$, by Lemma 3.4. Finally, let $\mathcal{V}' = \bigcup_{i\in\omega} \mathcal{V}_i$ and $T' = \bigcup_{i\in\omega} T_i'$.

$T'$ is complete: for any $\mathcal{V}'$-sentence $\varphi$, since $\varphi$ is finite, $\varphi$ is a $\mathcal{V}_i$-sentence for some $i \in \omega$, so $T_i' \vdash \varphi$ or $T_i' \vdash \neg\varphi$ in $\mathcal{V}_i$, since $T_i$ is a complete $\mathcal{V}_i$-theory. Since $T_i' \subseteq T'$, $T' \vdash \varphi$ or $T' \vdash \neg\varphi$ in $\mathcal{V}'$ by observation (2).

$T'$ is consistent: If $T' \vdash \bot$, then $\psi \vdash \bot$, where $\psi$ is a finite conjunction of sentences in $T'$. Since $\psi$ is finite, $\psi$ is a finite conjunction of sentences in $T_i'$ for some $i \in \omega$, and the proof is a proof in $\mathcal{V}_j$ for some $i \leq j \in \omega$, by observation (2). Then $\psi$ is also a finite conjunction of sentences in $T_j'$, and $T_j' \vdash \bot$ in $\mathcal{V}_j$, contradicting consistency of $T_j'$.

$T'$ has Henkin witnesses: If $T' \vdash \exists y\, \varphi(y)$, then as above we already have $T_j' \vdash \exists y\, \varphi(y)$ in $\mathcal{V}_j$ for some $j \in \omega$. Then there is a constant $c_\varphi$ in $\mathcal{V}_{k+1}$ such that $T_{k+1}' \vdash \varphi(c_\varphi)$ in $\mathcal{V}_{k+1}$, so $T' \vdash \varphi(c_\varphi)$ in $\mathcal{V}'$. □

**Theorem 3.10** (Completeness)**.** *If $T \models \varphi$, then $T \vdash \varphi$.*

*Proof.* Lemmas 3.7 and 3.9 together show that every consistent theory has a model. Indeed, if a $\mathcal{V}$-theory $T$ is consistent, then there is a vocabulary $\mathcal{V}'$ extending $\mathcal{V}$ by new constant symbols and a complete $\mathcal{V}'$-theory $T'$ with Henkin witnesses such that $T \subseteq T'$. Then $M(T') \models T'$. Since $T \subseteq T'$, $M(T') \models T$. And letting $M$ be the **reduct** of $M(T')$ to $\mathcal{V}$ (this just means we forget about the extra constant symbols in $\mathcal{V}'$, which aren't mentioned in $T$), we have $M \models T$.

Now we prove the theorem. If $T \models \varphi$, then every model of $T$ satisfies $\varphi$, i.e. $T \cup \{\neg\varphi\}$ has no models. Since every consistent theory has a model, it follows that $T \cup \{\neg\varphi\}$ is inconsistent, so there is a finite conjunction $\psi$ of sentences in $T$ such that $\psi \wedge \neg\varphi \vdash \bot$. By Lemma 2.5, $\psi \vdash \varphi$, so $T \vdash \varphi$. □

**Corollary 3.11** (Compactness)**.** *Let $T$ be a theory such that every finite subset of $T$ is satisfiable. Then $T$ is satisfiable.*

*Proof.* Suppose for contradiction that $T$ is not satisfiable, i.e. $T \models \bot$. By completeness, $T \vdash \bot$, so there is a finite subset $\Sigma \subseteq T$ such that $\bigwedge_{\psi\in\Sigma} \psi \vdash \bot$. By soundness, $\bigwedge_{\psi\in\Sigma} \psi \models \bot$, so $\Sigma \models \bot$, contradiction. □

The reason for the name is that the compactness theorem is a translation (using the completeness theorem) of the topological compactness of the Stone space $S_{()}$ of $\mathcal{L}_{()}$, the Boolean algebra of sentences. Precisely, given a theory $T$, each sentence $\varphi \in T$ corresponds to a clopen set $[\varphi] \subseteq S_{()}$. The hypothesis that every finite subset of $T$ is satisfiable tells us that for any finite $\Sigma \subseteq_{\text{fin}} T$, the clopen set $\bigcap_{\varphi\in\Sigma}[\varphi]$ is nonempty. Indeed, this intersection contains the ultrafilter on $\mathcal{L}_{()}$ corresponding to the complete theory of any model of $\Sigma$. Topological compactness of $S_{()}$ says that as a consequence, the closed set $\bigcap_{\varphi\in T}[\varphi]$ is nonempty. A point in the intersection corresponds to a complete consistent theory existending $T$, which has a model by completeness.

## 3.3 First applications of compactness

**Example 3.12.** The vocabulary of arithmetic is $\mathcal{V} = \{\leq, 0, 1, +, \times\}$. Consider the $\mathcal{V}$-structure $\mathbb{N} = (\mathbb{N}; \leq, 0, 1, +, \times)$. The complete theory $\mathrm{Th}(\mathbb{N})$ is called **true arithmetic**, and its model $\mathbb{N}$ is the **standard model** of arithmetic. Let's use compactness to show that $\mathrm{Th}(\mathbb{N})$ has *nonstandard* models.

Let $\mathcal{V}' = \mathcal{V} \cup \{c\}$, where $c$ is a new constant symbol. For each $n \in \mathbb{N}$, let $t_n$ be the term $\underbrace{1 + 1 + \cdots + 1}_{n \text{ times}}$. Define $T' = \mathrm{Th}(\mathbb{N}) \cup \{c \neq t_n \mid n \in \mathbb{N}\}$.

By compactness, so show that $T'$ is satisfiable, it suffices to show that any finite subset is satisfiable. For any finite subset $\Sigma \subseteq_{\text{fin}} T'$, pick some $n \in \mathbb{N}$ such that $(c \neq t_n) \notin \Sigma$. Then $(\mathbb{N}; n) \models \Sigma$, where the notation $(\mathbb{N}; n)$ means that we take $(\mathbb{N}; \leq, 0, 1, +, \times)$ and additionally interpret the constant symbol $c$ as the element $n$.

It follows that $T'$ is satisfiable, i.e. it has a model $(\mathcal{N}; \leq, 0, 1, +, \times, c)$. This model contains **standard** elements of the form $t_n^{\mathcal{N}}$ for all $n \in \mathbb{N}$, but it also contains the element $c^{\mathcal{N}}$, which is not equal to any standard element. It follows that $\mathcal{N}$ is not isomorphic to $\mathbb{N}$, despite the fact that it satisfies all of the same $\mathcal{V}$-sentences.

**Exercise 11.** Let $\mathcal{N}$ be a nonstandard model of $\mathrm{Th}(\mathbb{N})$ (that is, $\mathcal{N} \models \mathrm{Th}(\mathbb{N})$, but $\mathcal{N} \not\cong \mathbb{N}$).

(1) Show that the map $n \mapsto t_n^{\mathcal{N}}$ is an embedding $\mathbb{N} \to \mathcal{N}$. We will identify $\mathbb{N}$ with its image under this embedding.

(2) Show that any nonstandard element $a \in \mathcal{N} \setminus \mathbb{N}$ is greater than all standard elements, i.e. $\mathcal{N} \models n \leq a$ for all $n \in \mathbb{N}$.

(3) Imprecise question: what does the ordering of the nonstandard elements under $\leq$ look like? Is there a greatest nonstandard element? A least one? Are there infinitely many? Is their ordering dense? etc. etc.

(4) Show that $\mathcal{N}$ contains nonstandard "prime numbers". *Hint:* Show that for any $\mathcal{V}$-formula $\varphi(x)$ in a single free variable $x$, if there are arbitrarily large elements of $\mathbb{N}$ satisfying $\varphi(x)$, then there are nonstandard elements in $\mathcal{N}$ satisfying $\varphi(x)$.

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The compactness theorem is also a useful tool for showing that certain classes of structures cannot be axiomatized by first-order theories.

**Example 3.13.** The vocabulary of graphs is single-sorted and consists of a single binary relation symbol $E$. We will show that there is no theory $T$ such that $G \models T$ if and only if $G$ is a connected graph.

Suppose for contradiction that such a theory $T$ exists. Let $\mathcal{V}'$ be the vocabulary obtained by adding two new constant symbols $c$ and $d$ to the vocabulary of graphs, and let $T' = T \cup \{\varphi_n \mid n \in \omega\}$, where $\varphi_0$ is the sentence $(c \neq d)$, $\varphi_1$ is the sentence $\neg(cEd)$, and $\varphi_n$ is the sentence expressing that there is no path of length $n$ from $c$ to $d$:

$$\neg\exists x_1 \ldots \exists x_{n-1} \left( cEx_1 \wedge x_{n-1}Ed \wedge \bigwedge_{i=1}^{n-2} x_iEx_{i+1} \right).$$

We will show that $T'$ is consistent. For every natural number $n$, let $G_n$ be a connected graph containing elements $a_n$ and $b_n$ such that the length of the shortest path from $a_n$ to $b_n$ is $n$ (for example, we can take $G_n$ to consist of a path from $a_n$ to $b_n$ of length $n$).

Now any finite subset of $T'$ is contained in $T \cup \{\varphi_k \mid k < n\}$ for some large enough $n$, and $(G_n; a_n, b_n) \models T \cup \{\varphi_k \mid k < n\}$. So by compactness, $T'$ is consistent.

Let $(G; a, b)$ be a model of $T'$. Since $G \models T$, $G$ is a connected graph, but since $(G; a, b) \models \varphi_n$ for all $n$, there is no path of any length from $a$ to $b$. This is a contradiction.

We will later reframe this trick of adding constant symbols as the method of realizing types. You can use a similar idea to solve the following exercise.

**Exercise 12.** Work in the vocabulary of rings, $(0, 1, +, -, \cdot)$. Show that there is a theory $T$ such that $K \models T$ if and only if $K$ is a field of characteristic 0. In contrast, show that there is no theory $T$ such that $K \models T$ if and only if $K$ is isomorphic to an algebraic extension field of $\mathbb{Q}$.

# 4 Elementary embeddings

## 4.1 Homomorphisms and embeddings (again)

In this section, we fix a language $\mathcal{L}$, obtained from the vocabulary $\mathcal{V}$. We originally defined homomorphisms and embeddings just after defining structures. Let's return to them and think about their relationship with terms and formulas. The first observation is that homomorphisms respect the evaluation of complex terms.

**Proposition 4.1.** *Let $h\colon A \to B$ be a homomorphism. Then for any term $t$ in context $x$ and any interpretation $a \in A^x$, we have $h(t^A(a)) = t^B(h(a))$.*[12]

*Proof.* An easy induction on the complexity of terms. $\square$

For any formula $\varphi(x)$, we say that an $\mathcal{S}$-indexed map $h\colon A \to B$ **preserves** $\varphi(x)$ if for any $a \in A^x$, if $A \models \varphi(a)$, then $B \models \varphi(h(a))$. And we say that $h$ **reflects** $\varphi(x)$ if for any $a \in A^x$, if $B \models \varphi(h(a))$, then $A \models \varphi(a)$. Note that $h$ preserves $\varphi(x)$ if and only if it reflects $\neg\varphi(x)$.

**Proposition 4.2.** *Let $h\colon A \to B$ be an $\mathcal{S}$-indexed map.*

*(1) $h$ is a homomorphism if and only if it preserves all atomic formulas.*

*(2) $h$ is an embedding if and only if it preserves and reflects all atomic formulas (equivalently, it preserves all atomic and negated atomic formulas).*

*Proof.* Suppose $h$ preserves all atomic formulas. Then for every function symbol $f$, let $\varphi(x, y)$ be the atomic formula $f(x) = y$. If $f^A(a) = b$, then $A \models \varphi(a, b)$, so $B \models \varphi(h(a), h(b))$, and $f^B(h(a)) = h(b) = h(f^A(a))$.

Similarly, for every relation symbol $R$, let $\varphi(x)$ be the atomic formula $R(x)$. If $a \in R^A$, then $A \models \varphi(a)$, so $B \models \varphi(h(a))$, and $h(a) \in R^B$.

We have shown that $h$ is a homomorphism. If, moreover, $h$ reflects all atomic formulas, then the argument above shows that $a \in R^A$ if and only if $h(a) \in R^A$ for every relation symbol $R$. And for all $a, a' \in A$, if $h(a) = h(a)'$, then $B \models \varphi(a, a')$, where $\varphi(x, y)$ is the atomic formula $x = y$, so $A \models \varphi(a, a')$, and $a = a'$. It follows that $h$ is injective, so $h$ is an embedding.

Now let $\varphi(x)$ be an atomic formula, and $a \in A^x$.

*Case 1:* $\varphi$ is $t = u$. Suppose $h$ is a homomorphism. If $A \models \varphi(a)$, then $t^A(a) = u^A(a)$, so $h(t^A(a)) = h(u^A(a))$, and $t^B(h(a)) = u^B(h(a))$ by Proposition 4.1, so $B \models \varphi(h(a))$. If $h$ is an embedding, then the implications above is an equivalence, using the fact that $h$ is injective.

*Case 2:* $\varphi$ is $R(t_1, \ldots, t_n)$. Suppose $h$ is a homomorphism. If $A \models \varphi(a)$, then $(t_i^A(a))_{i=1}^n \in R^A$, so $(h(t_i^A(a)))_{i=1}^n \in R^B$, and $(t_i^B(h(a)))_{i=1}^n \in R^B$ by Proposition 4.1, so $B \models \varphi(h(a))$. If $h$ is an embedding, then the implications above is an equivalence. $\square$

[12] Here the context $x = (x_1, \ldots, x_n)$ is a tuple of variables, and each $x_i$ has a type $s_i$. Then the interpretation $a = (a_1, \ldots, a_n)$ is a tuple from $A$, such that each $a_i \in A_{s_i}$. Formally, we should write $h_{(s_1,\ldots,s_n)}(a)$ for the image of $a$ under the induced map $A_{(s_1,\ldots,s_n)} \to B_{(s_1,\ldots,s_n)}$. But from now on we will default to the simpler notation $h(a)$.

It is often useful to note that the class of structures which admit a homomorphism or an embedding from a structure $A$ are the models of a particular theory.

Given a $\mathcal{V}$-structure $A$ and a subset $B \subseteq A$, let $\mathcal{V}(B)$ be the vocabulary obtained from $\mathcal{V}$ by adding a new constant symbol of type $s$ for every element $b \in B_s$. When there is no chance for confusion, we will also denote the constant symbol by $b$. We view $A$ as a $\mathcal{V}(B)$ structure in the obvious way. In particular, $\mathcal{V}(A)$ is obtained by naming every element of $A$ by a constant symbol.

The **positive diagram** of a $\mathcal{V}$-structure $A$, denoted $\mathrm{Diag}^+(A)$, is the set of all atomic $\mathcal{V}(A)$-sentences true in $A$. That is, for every atomic $\mathcal{V}$-formula $\varphi(x)$ and every $a \in A^x$ such that $A \models \varphi(a)$, the $\mathcal{V}(A)$-sentence $\varphi(a)$ is in $\mathrm{Diag}^+(A)$.

Similarly, the **diagram** of $A$, denoted $\mathrm{Diag}(A)$, is the set of all atomic and negated atomic $\mathcal{V}(A)$-sentences true in $A$.

The following proposition now follows immediately from Proposition 4.2.

**Proposition 4.3.** *Let $A$ be a $\mathcal{V}$-structure, and let $B$ be a $\mathcal{V}(A)$-structure.*

1. $B \models \mathrm{Diag}^+(A)$ *if and only if the map $a \mapsto a^B$ is a homomorphism $A \to B$.*
2. $B \models \mathrm{Diag}(A)$ *if and only if the map $a \mapsto a^B$ is an embedding $A \to B$.*

## 4.2 Elementary equivalence and elementary embeddings

Structures $M$ and $N$ are **elementarily equivalent**, written $M \equiv N$, if for all sentences $\varphi \in \mathcal{L}_{()}$,

$$M \models \varphi \quad \text{iff} \quad N \models \varphi.$$

Equivalently, $\mathrm{Th}(M) = \mathrm{Th}(N)$.

An embedding $h\colon M \to N$ is an **elementary embedding** if it preserves (and reflects) all formulas. That is, for all formulas $\varphi(x) \in \mathcal{L}_x$ and all interpretations $a \in M^x$,

$$M \models \varphi(a) \quad \text{iff} \quad N \models \varphi(h(a)).$$

Of course, if $h$ preserves all formulas, then in particular it preserves the negation of every formula, so it also reflects every formula.

In particular, elementary embeddings preserve and reflect all sentences, so if $h\colon M \to N$ is an elementary embedding, then $M \equiv N$.

If $M \subseteq N$ and the inclusion is an elementary embedding, we write $M \prec N$, and say that $M$ is an **elementary substructure** of $N$, or $N$ is an **elementary extension** of $M$.

The $\mathcal{V}(M)$-theory $\mathrm{Th}_{\mathcal{V}(M)}(M)$ is called the **elementary diagram of** $M$, and denoted $\mathrm{EDiag}(M)$: it contains all of the information about all first-order formulas satisfied by all tuples from $M$. Similarly to Proposition 4.3, we have:

**Proposition 4.4.** *Let $M$ be a $\mathcal{V}$-structure, and let $N$ be a $\mathcal{V}(M)$-structure. Then $N \models \mathrm{EDiag}(M)$ if and only if the map $a \mapsto a^N$ is an elementary embedding $M \to N$.*

The following is a criterion for identifying elementary substructures. The advantage of this criterion is that condition (2) only refers to truth in the larger structure $N$.

**Theorem 4.5** (Tarski–Vaught Test). *Suppose $M$ is a substructure of $N$. The following are equivalent:*

*(1) $M \preceq N$.*

*(2) For every formula $\varphi(x, y)$ (where $x$ is a tuple of variables and $y$ is a single variable) and every tuple $a \in M^x$, if $N \models \exists y\, \varphi(a, y)$, then there is some element $b \in M$ such that $N \models \varphi(a, b)$.*

*Proof.* (1)⇒(2): Suppose $N \models \exists y\, \varphi(a, y)$. Since $a \in M^x$ and $M \preceq N$, $M \models \exists y\, \varphi(a, y)$. So there is some $b \in M$ such that $M \models \varphi(a, b)$, and since $M \preceq N$, also $N \models \varphi(a, b)$.

(2)⇒(1): We prove by induction on formulas $\varphi(x)$ that for all $a \in M^x$, $M \models \varphi(a)$ if and only if $N \models \varphi(a)$.

The base case, when $\varphi(x)$ is atomic, is handled by the fact that $M$ is a substructure of $N$. Then the inclusion $M \to N$ is an embedding, which preserves and reflects atomic formulas.

The inductive steps for Boolean combinations are straightforward. So we consider the case when $\varphi(x)$ is $\exists y\, \psi(x, y)$.

Assume $M \models \varphi(a)$. Then there is some $b \in M$ such that $M \models \psi(a, b)$. By induction $N \models \psi(a, b)$, so $N \models \varphi(a)$.

Conversely, assume $N \models \varphi(a)$, i.e. $N \models \exists y\, \psi(a, y)$. By (2), there is some $b \in M$ such that $N \models \psi(a, b)$. By induction, $M \models \psi(a, b)$, so $M \models \varphi(a)$. □

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The following easy lemma will also be useful sometimes.

**Lemma 4.6.** *Suppose $f : M_1 \to M_2$ and $g : M_2 \to M_3$ are embeddings. If $g$ and $g \circ f$ are both elementary embeddings, then so is $f$.*

*Proof.* For any $\varphi(x) \in \mathcal{L}_x$ and any $a \in M_1^x$, $M_1 \models \varphi(a)$ iff $M_3 \models \varphi(g(f(a)))$ iff $M_2 \models \varphi(f(a))$. □

We have now characterized three types of maps, based on the formulas they preserve: homomorphisms preserve atomic formulas, embeddings preserve atomic and negated atomic formulas, and elementary embeddings preserve all formulas. By this template, you can define many other types of maps between structures, by requiring that they preserve other sets of formulas. The strongest kinds of maps we will consider are the isomorphisms.

**Proposition 4.7.** *Every isomorphism is an elementary embedding.*

*Proof.* Let $h : M \to N$ be an isomorphism. We prove by induction on formulas $\varphi(x)$ that for all $a \in M^x$, $M \models \varphi(a)$ if and only if $N \models \varphi(h(a))$. Just as in the proof of Theorem 4.5, the base case of atomic formulas is handled by the fact that isomorphisms are embeddings, and the inductive steps for Boolean combinations are straightforward. So we consider the case when $\varphi(x)$ is $\exists y\, \psi(x, y)$. If $M \models \varphi(x)$, then there is some $b$ such that $M \models \psi(a, b)$, and by induction $N \models \psi(h(a), h(b))$, so $N \models \varphi(h(a))$. Conversely, if $N \models \varphi(h(a))$, there is some $b$ such that $N \models \psi(h(a), b)$, so by induction $M \models \psi(a, h^{-1}(b))$, and $M \models \varphi(a)$. $\square$

It follows that isomorphic structures are elementarily equivalent. We have already seen that the converse is not true in general (Example 3.12), but it is true for some structures, as discussed in the next section. The following example shows that not every embedding between elementarily equivalent structures is an elementary embedding.

**Example 4.8.** Consider the structure $(\mathbb{N}; \leq)$. The map $h\colon \mathbb{N} \to \mathbb{N}$ given by $n \mapsto n+1$ is an embedding of $\mathbb{N}$ in itself, but it is not an elementary embedding. For example, letting $\varphi(x)$ be the formula $\forall y\, (x \leq y)$, we have $\mathbb{N} \models \varphi(0)$, but $\mathbb{N} \not\models \varphi(h(0))$.

## 4.3 Counting

We now want to answer the following question: Given a theory $T$, what are the possible cardinalities of models of $T$? To address this question, we will need to recall just a little bit of set theory. A good reference for the facts below is *Introduction to Set Theory* by Hrabáček and Jech.

- The **ordinals** are a linearly ordered number system extending the natural numbers, characterized by the following properties:
  - $0$ is an ordinal.
  - For every ordinal $\alpha$, there is a next largest ordinal $\alpha + 1$ (this is a **successor** ordinal).
  - For every set of ordinals $S$, there is a least upper bound $\sup S$ (this is called a **limit** ordinal).
  - Every ordinal is either $0$, a successor, or a nonzero limit.
- The ordinals begin

  $$0, 1, 2, \dots, \omega, \omega+1, \omega+2, \dots, \omega+\omega (= \omega\cdot 2), \dots, \omega\cdot 3, \dots, \omega\cdot\omega (= \omega^2), \dots$$

- Formally, an ordinal $\alpha$ is identified with the set of ordinals less than $\alpha$. So $0 = \emptyset$, $1 = \{0\}$, $2 = \{0, 1\}$, $\omega = \{0, 1, 2, \dots\}$, etc. And if $\alpha$ and $\beta$ are ordinals, $\alpha \in \beta$ if and only if $\alpha < \beta$.
- The linear order on the ordinals is a **well-order**: every nonempty set of ordinals has a least element. This allows us to do constructions and proofs by **transfinite induction**. In addition to the usual base case and successor step for induction on $\omega$, it is also necessary to prove a limit step: Given a limit ordinal $\gamma$, if some property holds for all $\alpha < \gamma$, then it holds for $\gamma$.

- The ordinals (like all sets) are divided into equivalence classes for the "same cardinality" equivalence relation: $|\alpha| = |\beta|$ if there is a bijection between $\alpha$ and $\beta$. A **cardinal** is an ordinal which is the least element of its equivalence class.

- The axiom of choice implies that every set is in bijection with an ordinal[13]. It follows that every set $X$ is in bijection with a unique cardinal. This cardinal is called the **cardinality** of $X$, denoted $|X|$[14].

- The finite cardinals are $0, 1, 2, \ldots$, i.e. the natural numbers. $\aleph_0 = \omega$ is the smallest infinite cardinal. $\aleph_1$ is the smallest uncountable cardinal. Then we have $\aleph_2, \aleph_3, \ldots, \aleph_\omega, \aleph_{\omega+1}, \ldots$. The infinite cardinals can be indexed by the ordinals: every cardinal is of the form $\aleph_\alpha$ where $\alpha$ is an ordinal.

- Many facts about infinite cardinals are independent from ZFC set theory. The most famous independent statement is the Continuum Hypothesis: $2^{\aleph_0} = \aleph_1$, where $2^{\aleph_0}$ is the cardinality of $\mathcal{P}(\omega)$, or of $\mathbb{R}$. Cantor proved that $\aleph_0 < 2^{\aleph_0}$.

- However, we will use the following theorems about cardinalities:

  – Given sets $X$ and $Y$, with $|X| = \kappa$ and $|Y| = \lambda$, $\kappa+\lambda$ is the cardinality of $|X \sqcup Y|$, and $\kappa \cdot \lambda$ is the cardinality of $X \times Y$. If $\kappa$ and $\lambda$ are both infinite, then $\kappa + \lambda = \kappa \cdot \lambda = \max(\kappa, \lambda)$.

  – if $(X_i)_{i \in I}$ is an indexed family of sets, then the cardinality of $\bigcup_{i\in I} X_i$ is bounded above by the maximum of $|I|$ and $|X_i|$ for all $i \in I$. Indeed, if $\kappa = \max_{i\in I} |X_i|$, then

$$\left|\bigcup_{i\in I} X_i\right| \leq \left|\bigsqcup_{i\in I} \kappa\right| = |I \times \kappa| = \max(|I|, \kappa).$$

  – If $X$ is infinite, then the cardinality of the set $X^{<\omega}$ of finite sequences from $X$ is equal to the cardinality of $X$. Indeed,

$$|X| \leq |X^{<\omega}| = \left|\bigcup_{n\in\omega} X^n\right| \leq \max(\aleph_0, \max_{n\in\omega} |X^n|) = \max(\aleph_0, |X|) = |X|.$$

Recall that $\mathcal{L}$ is the set of all first-order formulas (in a fixed vocabulary $\mathcal{V}$), in any variable context. By $|\mathcal{V}|$, we mean $|\mathcal{S} \cup \mathcal{F} \cup \mathcal{R}|$.

**Lemma 4.9.** $|\mathcal{L}| = \max(|\mathcal{V}|, \aleph_0)$

[13] In fact, it is equivalent. This form of the axiom of choice is called the well-ordering principle.

[14] If we were not willing to assume the axiom of choice (but we are!), it would still be possible to define the cardinality of $X$ as the equivalence class of $X$ under the "same cardinality" equivalence relation. But the class of cardinalities is quite badly behaved without choice; for example, it is not linearly ordered

*Proof.* First, note that $|\mathcal{L}| \geq |\mathcal{V}|$, since we can associate a distinct formula to each sort, function symbol, and relation symbol in $\mathcal{V}$. For a sort $s \in \mathcal{S}$, take $x = x$, where $x$ is a variable of type $s$. For a function symbol $f \in \mathcal{F}$, take $f(x_1, \ldots, x_n) = y$. For a relation symbol $R \in \mathcal{R}$, take $R(x_1, \ldots, x_n)$.

Also, $|\mathcal{L}| \geq \aleph_0$, since there are trivially infinitely many formulas. For example, $\top$, $\neg\top$, $\neg\neg\top$, $\neg\neg\neg\top$, etc. is an infinite sequence of formulas.

So $|\mathcal{L}| \geq \max(|\mathcal{V}|, \aleph_0)$. Conversely, note that a formula is a finite sequence of symbols, which could be function symbols from $\mathcal{F}$, relation symbols from $\mathcal{R}$, variables from our supply $X = (X_s)_{s \in \mathcal{S}}$, or syntactic symbols from the set $\mathrm{Syn} = \{=, \top, \bot, \wedge, \vee, \neg, \exists, (, )\}$. Let $\mathrm{Symb}(\mathcal{L}) = \mathcal{F} \cup \mathcal{R} \cup X \cup \mathrm{Syn}$. Recall that each set $X_s$ is countably infinite, so $|X| = \left|\bigcup_{s \in \mathcal{S}} X_s\right| = \max(|\mathcal{S}|, \aleph_0)$. It follows that $|\mathrm{Symb}(\mathcal{L})| = \max(|\mathcal{F}|, |\mathcal{R}|, |\mathcal{S}|, \aleph_0) = \max(|\mathcal{V}|, \aleph_0)$, since if $\mathcal{V}$ is infinite, then $|\mathcal{V}| = \max(|\mathcal{F}|, |\mathcal{R}|, |\mathcal{S}|)$, while if $\mathcal{V}$ is finite, both sides are $\aleph_0$.

Now since every formula is a finite sequence from $\mathrm{Symb}(\mathcal{L})$:

$$|\mathcal{L}| \leq |\mathrm{Symb}(\mathcal{L})|^{<\omega} = |\mathrm{Symb}(\mathcal{L})| = \max(\mathcal{V}, \aleph_0). \qquad \square$$

## 4.4 Boring and interesting theories

A $\mathcal{V}$-structure $A$ is **boring** if $A_s$ is finite for all $s \in \mathcal{S}$. Otherwise, it is **interesting**.[15] A theory $T$ is **boring** if all of its models are boring. On the other hand, $T$ is **interesting** if it has an interesting model. In particular, all interesting theories are consistent.

Recall that we defined the cardinality of an $\mathcal{S}$-indexed set $A$ to be the cardinality of the disjoint union of its components: $|A| = |\bigsqcup_{s \in \mathcal{S}} A_s|$. As usual, we say that $A$ is finite when $|A| = n$ for some $n \in \mathbb{N}$. When the vocabulary $\mathcal{V}$ has only finitely many sorts, a structure is boring if and only if it is finite. But when $\mathcal{V}$ has infinitely many sorts, there are infinite boring structures.

**Proposition 4.10.** *Assume $\mathcal{V}$ is a finite vocabulary. If $A$ is a boring $\mathcal{V}$-structure, then there is a sentence $\varphi_A$ such that $B \models \varphi_A$ if and only if $B \cong A$.*

*Proof.* Since $\mathcal{V}$ has only finitely many sorts and $A$ is boring, $A$ is actually finite. Let $a_1, \ldots, a_n$ be an enumeration of $A$, and let $x_i$ be a variable of type $s$ corresponding to each element $a_i \in A_s$.

First, we write down a formula expressing that the variables $x_i$ enumerate the structure. Let $\chi(x_1, \ldots, x_n)$ be the formula

$$\left(\bigwedge_{1 \leq i < j \leq n} x_i \neq x_j\right) \wedge \left(\forall y \bigvee_{1 \leq i \leq n} y = x_i\right).$$

Now we write down the interpretations of all function and relation symbols in $\mathcal{V}$. For each function symbol $f$, let $\psi_f(x_1, \ldots, x_n)$ be the conjunction of all formulas of the form $f(x_{i_1}, \ldots, x_{i_k}) = x_j$ such that $A \models f(a_{i_1}, \ldots, a_{i_k}) = a_j$. And for each relation symbol $R$, let $\psi_R(x_1, \ldots, x_n)$ be the conjunction of all formulas

[15] This terminology is not standard in model theory.

of the form $R(x_{i_1}, \dots, x_{i_k})$ or $\neg R(x_{i_1}, \dots, x_{i_k})$ such that $A \models R(a_{i_1}, \dots, a_{i_k})$ or $A \models \neg R(a_{i_1}, \dots, a_{i_k})$, respectively. These are finite conjunctions, since $A$ is finite.

Then define $\varphi_A$ to be

$$\exists x_1 \dots \exists x_n \left( \chi(x_1, \dots, x_n) \wedge \left( \bigwedge_{f \in \mathcal{F}} \psi_f(x_1, \dots, x_n) \right) \wedge \left( \bigwedge_{R \in \mathcal{R}} \psi_R(x_1, \dots, x_n) \right) \right).$$

The conjunctions in $\varphi_A$ are finite since $\mathcal{V}$ is finite.

Now if $B \models \varphi_A$, then letting $b_1, \dots, b_n$ be witnesses for the existential quantifiers, it is clear that the map $a_i \mapsto b_i$ is an isomorphism $A \to B$. $\square$

Thursday 9/27 **Exercise 13.** Remove the finiteness hypothesis from Proposition 4.10, as follows: Let $\mathcal{V}$ be any vocabulary. If $A$ is a boring $\mathcal{V}$-structure, then for any $\mathcal{V}$-structure $B$, $B \equiv A$ implies $B \cong A$. (*Suggestions:* First try to prove this when $\mathcal{V}$ has only finitely many sorts, but an arbitrary set of function and relation symbols. Then extend to the case with infinitely many sorts.)

The upshot is that first-order logic is powerful enough to pin down boring structures completely up to isomorphism. Ironically, this means that we will be almost totally uninterested in boring structures and boring theories. Much of the interest in model theory comes from working with structures in which not everything is definable, and moving between distinct models of a theory.

## 4.5 The Löwenheim–Skolem theorems

We are finally ready to show that there are non-trivial elementary embeddings (i.e. elementary embeddings which are not isomorphisms), and to address the question of possible cardinalities of models of $T$ (at least for interesting $T$).

**Theorem 4.11** (Downwards Löwenheim–Skolem)**.** *Suppose $M$ is a structure and $A \subseteq M$. Then there is an elementary substructure $N \preceq M$ such that $A \subseteq N$ and $|N| \leq \max(|A|, |\mathcal{L}|)$.*

*Proof.* We define a sequence of subsets of $M$, $(A_i)_{i \in \omega}$, by induction. Let $A_0 = A$.

Given $A_i$, for every formula $\varphi(x, y)$ (where $x$ is a tuple of variables and $y$ is a single variable) and every $a \in A_i^x$, if $M \models \exists y\, \varphi(a, y)$, pick some element $b_{\varphi(a,y)} \in M$ such that $M \models \varphi(a, b)$. Define

$$A_{i+1} = A_i \cup \{b_{\varphi(a,y)} \mid \varphi(x, y) \in \mathcal{L}_{xy}, a \in A_i^x\}.$$

Finally, let $N = \bigcup_{i \in \omega} A_i$.

We want to view $N$ as a substructure of $M$. To do this, we need to check that $N$ is closed under the interpretations of all the function symbols in $\mathcal{V}$. So suppose $f$ is a function symbol of type $(s_1, \dots, s_n) \to s$, and let $a \in N_{(s_1, \dots, s_n)}$. Since $a$ is a finite tuple, there is already some $i \in \omega$ such that $a \in (A_i)_{(s_1, \dots, s_n)}$. Then $M \models \exists y\, (f(a) = y)$, so there is some $b \in A_{i+1}$ such that $M \models (f(a) = b)$. So $f^M(a) = b \in N$.

Next, we show that $N \preceq M$ using the Tarski–Vaught test, Theorem 4.5. So suppose $\varphi(x, y)$ is a formula and $a \in N^x$, such that $M \models \exists y\, \varphi(a, y)$. Then there is already some $i \in \omega$ such that $a \in A_i^x$, and by construction there is some $b \in A_{i+1} \subseteq N$ such that $M \models \varphi(a, b)$, as was to be shown.

It remains to show the bound on the cardinality of $N$. Let $\kappa = \max(|A|, |\mathcal{L}|)$. We show by induction that $|A_i| \leq \kappa$ for all $i \in \omega$. In the base case, $A_0 = A$, and the inequality is clear. So we consider $A_{i+1}$. For any formula $\varphi(x, y)$, where $x$ is a tuple of length $n$, the set $B_{\varphi(x,y)} = \{b_{\varphi(a,y)} \mid a \in A_i^x\}$ has cardinality at most $|A_i|^n$. This is equal to $|A_i|$ when $A_i$ is infinite, and it is finite when $A_i$ is finite. So in either case, $|B_{\varphi(x,y)}| \leq \max(|A_i|, \aleph_0) \leq \kappa$ by induction, and since $\kappa$ is infinite.

Now by induction again, and since $|\mathcal{L}| \leq \kappa$:

$$\begin{aligned} |A_{i+1}| &= \left| A_i \cup \bigcup_{\varphi(x,y) \in \mathcal{L}} B_{\varphi(x,y)} \right| \\ &\leq |A_i| + \max(|\mathcal{L}|, \kappa) \\ &\leq \kappa. \end{aligned}$$

Finally, we have $|N| = |\bigcup_{i \in \omega} A_i| \leq \max(\aleph_0, \kappa) = \kappa$. □

Theorem 4.11 was the source of Skolem's "paradox". In modern language: ZFC set theory proves that there are uncountably infinite sets (like $\mathcal{P}(\omega)$). But if ZFC is consistent, then it has a countably infinite model. How can a countably infinite model of set theory contain uncountably infinite sets? The resolution of the paradox is that if $M$ is a countable model of ZFC, then working *outside* the model, we can put $M$ in bijection with the *real* natural numbers. But *in* $M$, there is no element of $M$ which is a bijection between the element of $M$ called $\mathcal{P}(\omega)$ and the element of $M$ called $\omega$.

**Theorem 4.12** (Upwards Löwenheim–Skolem)**.** *Suppose $M$ is an interesting structure and $\kappa$ is a cardinal with $\kappa \geq \max(|M|, |\mathcal{L}|)$. Then there is an elementary extension $M \preceq N$ with $|N| = \kappa$.*

*Proof.* Recall that $\mathcal{V}(M)$ is the vocabulary obtained by adding new constant symbol to $\mathcal{V}$ naming every element of $M$. To find an elementary extension of $M$, we will find a model of the $\mathcal{V}(M)$-theory $\mathrm{EDiag}(M)$.

Since $M$ is interesting, there is some sort $s \in \mathcal{S}$ such that $M_s$ is infinite. Let $\mathcal{V}' = \mathcal{V}(M) \cup \{c_\alpha \mid \alpha < \kappa\}$ be the vocabulary obtained by adding $\kappa$ many new constant symbols of type $s$ to $\mathcal{V}(M)$.

Let $T' = \mathrm{EDiag}(M) \cup \{c_\alpha \neq c_\beta \mid \alpha < \beta < \kappa\}$. We will show by compactness that $T'$ is consistent. A finite subset of $T'$ is contained in the theory $\mathrm{EDiag}(M) \cup \{c_{\alpha_i} \neq c_{\alpha_j} \mid i \neq j\}$ for some finitely many $\alpha_1 < \cdots < \alpha_n < \kappa$. Since $M_s$ is infinite, we can pick $n$ distinct elements $a_1, \ldots, a_n \in M_s$. Then interpreting the constant $c_{\alpha_i}$ as $a_i$, $M$ itself is a model of this finite subset of $T'$.

Now let $N'$ be a model of $T'$. Then by Proposition 4.4, the map $a \mapsto a^N$ is an elementary embedding $M \to N'$. Identifying $M$ with its image under this map, we may assume $M \preceq N'$. And $|N'| \geq \kappa$, since the elements $(c_\alpha^{N'})_{\alpha<\kappa}$ are all distinct.

To get an elementary extension of cardinality exactly $\kappa$, let $A = M \cup \{c_\alpha^{N'} \mid \alpha < \kappa\}$, so $|A| = \kappa$. By Theorem 4.11, there is an elementary substructure $N \preceq N'$ such that $M \subseteq A \subseteq N$ and $|N| \leq \max(|A|, |\mathcal{L}|) = \kappa$. Also $A \subseteq N$, so $|N| = \kappa$. And $M \preceq N$ by Lemma 4.6. □

**Corollary 4.13.** *Let $T$ be an interesting theory, and suppose $\kappa \geq |\mathcal{L}|$. Then $T$ has a model of cardinality $\kappa$.*

*Proof.* Let $M$ be an interesting model of $T$. If $|M| = \kappa$, we are done.

If $|\mathcal{L}| \leq \kappa < |M|$, let $A$ be an arbitrary subset of $M$ of cardinality $\kappa$. By Theorem 4.11, there exists $N \preceq M$ such that $|N| \leq \max(|A|, |\mathcal{L}|) = \kappa$, and since $A \subseteq N$, also $\kappa \leq |N|$. So $N$ is model of $T$ of cardinality $\kappa$.

If $\kappa > |M|$, then since $\kappa \geq |\mathcal{L}|$, we have $\max(|M|, \mathcal{L}) \leq \kappa$. By Theorem 4.12, there exists $N$ of cardinality $\kappa$ with $M \preceq N$, so $N \models T$. □

Note that the Löwenheim–Skoelm theorems do not guarantee existence of models of cardinality $\kappa$ when $\kappa < |\mathcal{L}|$. Model theory has very little to say in general about models which are smaller than the cardinality of the language.

## 4.6 $\kappa$-categoricity and completeness

We have shown that every interesting theory has a proper class of models up to isomorphism, at least one of every cardinality $\kappa \geq |\mathcal{L}|$. In particular (using Exercise 13), the theories which have just a single model up to isomorphism are exactly the complete boring theories.

This suggests an interesting question: If you fix an infinite cardinal $\kappa$, are there theories which have exactly one model of cardinality $\kappa$ up to isomorphism? The answer is yes: such theories are called **$\kappa$-categorical**.

**Proposition 4.14** (Vaught's Test)**.** *Suppose $T$ is a consistent theory with no boring models, and that $T$ is $\kappa$-categorical for some $\kappa \geq |\mathcal{L}|$. Then $T$ is complete.*

*Proof.* Suppose for contradiction that $T$ is not complete. Then there is some sentence $\varphi$ such that $T_1 = T \cup \{\varphi\}$ and $T_2 = T \cup \{\neg\varphi\}$ are both consistent. Since $T$ has no boring models, both $T_1$ and $T_2$ are interesting. By Corollary 4.13, there are models $M_1 \models T_1$ and $M_2 \models T_2$ such that $|M_1| = |M_2| = \kappa$. But then by $\kappa$-categoricity, $M_1 \cong M_2$. This contradicts the fact that $M_1 \models \varphi$ and $M_2 \models \neg\varphi$. □

Vaught's test gives us our first really explicit examples of complete theories.

**Example 4.15.** Let $\mathcal{V}_\emptyset$ be the vocabulary with a single sort and no function or relation symbols. A $\mathcal{V}_\emptyset$-structure is just a set. For every $n$, let $\varphi_n$ be the sentence

$$\exists x_1 \ldots \exists x_n \bigwedge_{1 \leq i < j \leq n} x_i \neq x_j.$$

Then $A \models \varphi_n$ if and only if $|A| \geq n$. Let $T_\infty = \{\varphi_n \mid n \in \omega\}$. The models of $T_\infty$ are exactly the infinite sets. $T_\infty$ is $\kappa$-categorical for every infinite cardinal $\kappa$, since an isomorphism of $\mathcal{V}_\emptyset$-structures is just a bijection, and there is a bijection between any two sets of cardinality $\kappa$. Since also $T_\infty$ has no boring (finite) models, $T_\infty$ is complete by Vaught's test.

**Example 4.16.** Consider the vocabulary of strict orders, $(<)$. Let DLO be the theory of (non-empty) dense linear orders without endpoints. This is axiomatized by the following sentences:

- $\forall x\, \neg(x < x)$.
- $\forall x\, \forall y\, \forall z\, ((x < y \wedge y < z) \to x < z)$.
- $\forall x\, \forall y\, (x < y \vee y < x \vee x = y)$.
- $\exists x\, \top$.
- $\forall x\, \forall y\, (x < y \to \exists z\, (x < z \wedge z < y))$.
- $\forall x\, \exists y\, x < y$.
- $\forall x\, \exists y\, y < x$.

Note that $(\mathbb{Q}, <) \models \mathrm{DLO}$. It is a theorem of Cantor (which we will prove below) that DLO is $\aleph_0$-categorical. Also, no model of DLO is finite, since any nonempty finite linear order has a maximum element. So DLO is complete by Vaught's test.

We will now prove Cantor's classical theorem on dense linear orders without endpoints. This theorem, from 1895, predates everything else in these notes (Gödel's completeness theorem is from 1929, and model theory did not really become an independent branch of logic until the 1950s). We will give a modern proof using the "back-and-forth" method, which is an important technique in model theory.

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**Theorem 4.17.** *Any two countable dense linear orders without endpoints are isomorphic.*

*Proof.* Let $M$ and $N$ be countable models of DLO. Since they are countably infinite, we can enumerate them as $M = (m_i)_{i\in\omega}$ and $N = (n_i)_{i\in\omega}$, respectively.

By induction, we will construct for all $i \in \omega$: a finite substructure $A_i \subseteq M$, a finite substructure $B_i \subseteq N$, and an isomorphism $f_i \colon A_i \to B_i$. In the course of the construction, we will ensure that when $i \leq j$, $A_i \subseteq A_j$, $B_i \subseteq B_j$, and $f_j|_{A_i} = f_i$. We will also ensure that for all $i \in \omega$, $m_i \in A_{i+1}$ and $n_i \in B_{i+1}$.

This suffices, since then $\bigcup_{i\in\omega} A_i = M$, $\bigcup_{i\in\omega} B_i = N$, and the coherent sequence of isomorphisms $(f_i)_{i\in\omega}$ induce an isomorphism $f \colon M \to N$.

In the base case, take $A_0 = B_0 = \emptyset$ and $f_0$ the empty function.

Given $f_i \colon A_i \to B_i$, we first go "forth". If $m_i \in A_i$, let $A_i^* = A_i$, $B_i^* = B_i$, and $f_i^* = f_i$. Otherwise, let $a_-$ be the greatest element of $A_i$ less than $m_i$

(or $-\infty$ if there is none), and let $a_+$ be the least element of $A_i$ greater than $m_i$ (or $+\infty$ if there is none). Since $f_i$ is an isomorphism $A_i \to B_i$, letting $f_i(-\infty) = -\infty$ and $f_i(+\infty) = +\infty$, we can find some element $n$ in the interval $(f_i(a_-), f_i(a_+))$ in $N$. This uses density, no greatest element, no least element, or non-emptyness, depending on whether $a_- = -\infty$ and whether $a_+ = +\infty$. Define $A_i^* = A_i \cup \{m_i\}$, $B_i^* = B_i \cup \{n\}$, and $f_i^*$ to be the map extending $f_i$ by $m_i \mapsto n$. Then $f_i^*$ is an isomorphism.

Given $f_i^* \colon A_i^* \to B_i^*$, we now go "back". If $n_i \in B_i^*$, let $A_{i+1} = A_i^*$, $B_{i+1} = B_i^*$, and $f_{i+1} = f_i^*$. Otherwise, just as above we find an element $m \in M$ such that the map $f_{i+1}$ extending $f_i^*$ by $m \mapsto n_i$ gives an isomorphism from $A_{i+1} = A_i^* \cup \{m\}$ to $B_{i+1} = B_i^* \cup \{n_i\}$. This completes the construction. □

The following exercise generalizes the back-and-forth method to give a useful criterion for isomorphism between countable structures and elementary equivalence between arbitrary structures.

**Exercise 14.** A **partial isomorphism** $M \dashrightarrow N$ is an isomorphism from a substructure of $M$ to a substructure of $N$. If $\sigma$ and $\tau$ are partial isomorphisms $M \dashrightarrow N$, we say $\tau$ **extends** $\sigma$ if $\mathrm{dom}(\sigma) \subseteq \mathrm{dom}(\tau)$ and $\sigma(a) = \tau(a)$ for all $a \in \mathrm{dom}(\sigma)$ (i.e. set theoretically, $\sigma \subseteq \tau$). A **back-and-forth system** between $M$ and $N$ is a non-empty family $P$ of partial isomorphisms $M \dashrightarrow N$ such that:

- For all $\sigma \in P$ and all $a \in M$, there exists $\tau \in P$ such that $\tau$ extends $\sigma$ and $a \in \mathrm{dom}(\tau)$.
- For all $\sigma \in P$ and all $b \in N$, there exists $\tau \in P$ such that $\tau$ extends $\sigma$ and $b \in \mathrm{ran}(\tau)$.

We say $M$ and $N$ are **partially isomorphic**, written $M \cong_p N$, if there is a back-and-forth system between $M$ and $N$.

(a) Show that if $M \cong N$, then $M \cong_p N$.

(b) Show that if $M$ and $N$ are countable and $M \cong_p N$, then $M \cong N$.

(c) Show that if $M \cong_p N$, then $M \equiv N$.

(d) Suppose $M$ is a substructure of $N$, and that there is a back-and-forth system $P$ between $M$ and $N$ such that for every finite $A \subseteq_{\mathrm{fin}} M$, the identity map $\langle A \rangle \to \langle A \rangle$, viewed as a partial isomorphism $M \dashrightarrow N$, is in $P$. Show that $M \preceq N$.

**Exercise 15.** Work in the vocabulary of strict orders ($<$). Show that $\mathbb{Q} \preceq \mathbb{R}$ (where both sets are equipped with their usual orders). *Hint:* Use Exercise 14(d), and consider the set $P$ of *all* partial isomorphisms $\sigma \colon \mathbb{Q} \dashrightarrow \mathbb{R}$ such that $\mathrm{dom}(\sigma)$ is finite.

**Example 4.18.** Consider the vocabulary of rings $(0, 1, +, -, \cdot)$. Let ACF be the theory of algebraically closed fields. This is axiomatized by the following sentences:

- The field axioms.
- The following sentence $\varphi_d$ for every degree $d \geq 1$:

$$\forall a_0 \ldots \forall a_{d-1} \exists y \, (y^d + a_{d-1} \cdot y^{d-1} + \cdots + a_1 \cdot y + a_0 = 0)$$

Of course, in the sentence $\varphi_d$, we are using some shorthands, like writing $y^n$ for $\underbrace{y \cdot \ldots \cdot y}_{n \text{ times}}$ and omitting the parentheses that show the order of the additions (associativity of addition is not built into the syntax, but rather follows from the other field axioms).

The theory ACF is not complete, but we can describe all of its completions. For any prime number $p$, we define $\mathrm{ACF}_p = \mathrm{ACF} \cup \{p = 0\}$, and we define $\mathrm{ACF}_0 = \mathrm{ACF} \cup \{p \neq 0 \mid p \text{ prime}\}$. Here $p$ is shorthand for the term $\underbrace{1 + \ldots + 1}_{p \text{ times}}$.

We will show below that the theories $\mathrm{ACF}_0$ and $\mathrm{ACF}_p$ for any prime $p$ are all $\kappa$-categorical for every uncountable cardinal $\kappa$. Since these theories also have no boring models (every algebraically closed field is infinite), it follows from Vaught's test that they are complete.

Note that $\mathrm{ACF}_0$ is not $\aleph_0$-categorical, since the fields $\overline{\mathbb{Q}(t_1, \ldots, t_n)}$ are all countably infinite but non-isomorphic for distinct $n$. Replacing $\mathbb{Q}$ with $\mathbb{F}_p$ shows that $\mathrm{ACF}_p$ is not $\aleph_0$-categorical.

To prove the following proposition, we'll use some field theory (a reference is Lang's *Algebra*, Chapter VIII). The relevant fact is that an algebraically closed field is determined up to isomorphism by its charactersitic and its transcendence degree over the prime field.

**Proposition 4.19.** *Let $p$ be a prime number or $0$, and let $\kappa$ be an uncountable cardinal. Then $\mathrm{ACF}_p$ is $\kappa$-categorical.*

*Proof.* Let $M \models \mathrm{ACF}_p$. Then $M$ contains an isomorphic copy $F$ of the prime field ($F \cong \mathbb{Q}$ if $p = 0$, $F \cong \mathbb{F}_p$ otherwise). We can factor the field extension $F \subseteq M$ as $F \subseteq K \subseteq M$, where $K = F(B)$ is a purely transcendental extension of $F$, generated by a transcendence basis $B$, and $M$ is the algebraic closure of $K$. Now some counting shows that $|M| = |K| = \max(|B|, \aleph_0)$.

It follows that if also $N \models \mathrm{ACF}_p$ with prime field $F'$ and transcendence basis $B'$, and $|M| = |N| = \kappa$, then we have $|B| = |B'| = \kappa$, since $\kappa$ is uncountable. So we can start with an isomorphism $F \to F'$, extend this to an isomorphism $F(B) \to F(B')$ by picking a bijection $B \to B'$, and extend this to an isomorphism $M \to N$ between the algebraic closures. □

**Corollary 4.20** (Transfer principle for algebraically closed fields)**.** *Let $\varphi$ be a sentence in the vocabulary of rings. The following are equivalent:*

*(1) Some algebraically closed field of characteristic $0$ satisfies $\varphi$.*

*(2) Every algebraically closed field of characteristic $0$ satisfies $\varphi$.*

*(3) For all but finitely many primes $p$, some algebraically closed field of characteristic $p$ satisfies $\varphi$.*

*(4) For all but finitely many primes $p$, every algebraically closed field of characteristic $p$ satisfies $\varphi$.*

*Proof.* (1)⇒(2): If $M \models \mathrm{ACF}_0$ and $M \models \varphi$, then $\mathrm{ACF}_0 \not\models \neg\varphi$, so $\mathrm{ACF}_0 \models \varphi$ by completeness.

(2)⇒(4): We have $\mathrm{ACF}_0 \models \varphi$, so by compactness there are finitely many primes $p_1, \dots, p_n$ such that $\mathrm{ACF} \cup \{p_i \neq 0 \mid 1 \leq i \leq n\} \models \varphi$. For any prime $q$ not among these finitely many exceptions, any algebraically closed field of characteristic $q$ satisfies $\mathrm{ACF} \cup \{p_i \neq 0 \mid 1 \leq i \leq n\}$, hence satisfies $\varphi$.

(4)⇒(3): Trivial.

(3)⇒(1): Assume for contradiction that (1) fails. Then every algebraically closed field of characteristic 0 satisfies $\neg\varphi$. By (2)⇒(4) for $\neg\varphi$, we have that for all but finitely many primes $p$, every algebraically closed field of characteristic $p$ satisfies $\neg\varphi$. This contradicts (3) (since there are infinitely many primes!). □

Thursday 10/4 **Exercise 16.** Fix a field $K$, and consider the vocabulary of $K$-vector spaces described in Example 1.2. Let $T$ be the theory of $K$-vector spaces.

(a) If $V$ is a $K$-vector space with a basis $B$, find $|V|$ in terms of $|K|$ and $|B|$.

(b) For which cardinals $\kappa$ is $T$ $\kappa$-categorical?

(c) $T$ is not a complete theory. Describe its completions.

In all parts above, you should consider carefully what happens when $K$ is finite.

Do not get the wrong impression from the examples above: $\kappa$-categoricity is a very unusual property for a theory (even a complete theory) to have! Later in this class, we will study $\aleph_0$-categorical theories in general. It turns out that $\aleph_0$-categorical theories have very special properties, which are quite different from the properties of $\kappa$-categorical theories for uncountable $\kappa$.

I will end this section by mentioning Morley's Categoricity Theorem:

**Theorem 4.21** (Morley 1965 for countable $\mathcal{L}$, Shelah 1974 for general $\mathcal{L}$)**.** *Suppose $\kappa > |\mathcal{L}|$ and $T$ is a complete $\kappa$-categorical theory. Then $T$ is $\lambda$-categorical for all $\lambda > |\mathcal{L}|$.*

The statement of Morley's theorem is interesting, in that it confirms a pattern we see in the examples above, but the real value in this theorem is the methods of proof, which inspired much of modern model theory. The proof (in a reformulation by Baldwin and Lachlan) shows that whenever $T$ is $\kappa$-categorical and $\kappa > |\mathcal{L}|$, there is some way of associating to every model of $T$ a kind of abstract "geometry" (more precisely, a *matroid*). This geometry comes with a notion of basis and dimension, and the isomorphism type of a model is completely determined by its dimension. This abstract dimension generalizes cardinality in the case of infinite sets, linear dimension in the case of $K$-vector spaces, and transcendence degree in the case of algebraically closed fields of fixed characteristic.

# 5 Definability

## 5.1 Formulas and types

Up to now, we have been primarily concerned with sentences and theories, which express properties of structures. We will now turn our eyes inward, fixing a theory $T$ of interest, and think about formulas and types, which express properties of (tuples of) elements from models of $T$.

So for the following discussion, we fix a vocabulary $\mathcal{V}$, the associated first-order language $\mathcal{L}$, and a (not necessarily complete) theory $T$.

The first observation is that if $M \models T$, there are certain special subsets of $M$ (and more generally of $M^x$), namely those which are defined by formulas. Given a formula $\varphi(x)$, we define

$$\varphi(M) = \{a \in M^x \mid M \models \varphi(a)\}.$$

This is called a **definable set** in $M$. More generally, given a formula $\varphi(x; y)$ (here the context $xy$ has been **partitioned** into a tuple of variables $x$ and a tuple of variables $y$), and any $b \in M^y$, we define

$$\varphi(M; b) = \{a \in M^x \mid M \models \varphi(a; b)\}.$$

This is called a **definable set (with parameters)** in $M$.

Further, given formulas $\varphi(x; y)$ and $\psi(y)$, we can look at the **definable family of definable sets** $\{\varphi(M; b) \mid b \in \psi(M)\}$.

The second observation is that a single formula $\varphi(x)$ gives a definable set in every model of $T$. And if $h\colon M \to N$ is an elementary embedding, then $h$ induces a function $\varphi(M) \to \varphi(N)$. And similarly, given $\varphi(x; y)$, $h$ induces a function $\varphi(M; b) \to \varphi(N; h(b))$. So it makes sense to consider definable sets not just as subsets of a fixed model, but varying over the whole category of models of $T$. We will fix our definitions in this more general setting.

We say that formulas $\varphi(x)$ and $\psi(x)$ in context $x$ are **$T$-equivalent** if

$$T \models \forall x\, (\varphi(x) \leftrightarrow \psi(x)).$$

That is, for any model $M \models T$, $\varphi(M) = \psi(M)$.

Here $x = (x_1, \ldots, x_n)$ may be a tuple of variables, in which case $\forall x$ is shorthand for $\forall x_1 \ldots \forall x_n$. If $x$ is the empty context (i.e. $\varphi$ and $\psi$ are sentences), then $\forall x$ is the empty sequence of quantifiers. We use similar shorthand for $\exists x$ when $x$ is a tuple of variables.

We write $\mathcal{L}_x(T)$ for the set of formulas in context $x$, modulo $T$-equivalence.

**Exercise 17.** Show that $\mathcal{L}_x(T)$ has a natural Boolean algebra structure. In fact, it is isomorphic to the quotient of the Boolean algebra $\widehat{\mathcal{L}_x}$ of formulas modulo logical equivalence defined in Remark 3.3, by the filter consisting of all (equivalence classes of) formulas $\varphi(x)$ such that $T \models \forall x\, \varphi(x)$ (see Exercise 9). For any $M \models T$, the map $\varphi(x) \mapsto \varphi(M)$ is a homomorphism $\mathcal{L}_x(T) \to \mathcal{P}(M^x)$. If $T$ is complete, this homomorphism is injective.

Having done the exercise, it will be clear to you that the logical operations $\wedge$, $\vee$, and $\neg$ correspond to intersection, union, and complement of definable sets in context $x$, respectively. The existential quantifier, on the other hand, corresponds to coordinate projection. That is, for any model $M$, context $x$, and new variable $y$, there is a coordinate projection map $\pi_x \colon M^{xy} \to M^x$, defined by $(a_1, \ldots, a_n, b) \mapsto (a_1, \ldots, a_n)$. Then for any formula $\varphi(x, y)$, we have $(\exists y\, \varphi)(M) = \pi_x(\varphi(M))$.

A **partial type** in context $x$ is a set of formulas in context $x$. We say that a partial type $\Sigma(x)$ is **realized** by $a \in M^x$, written $M \models \Sigma(a)$ if $M \models \varphi(a)$ for all $\varphi(x) \in \Sigma(x)$. We say that $\Sigma(x)$ is **satisfiable** if it is realized in some model $M \models T$. A **(complete) type** $p(x)$ is a satisfiable partial type such that for every formula $\varphi(x)$ in context $x$, either $\varphi(x) \in p(x)$, or $\neg\varphi(x) \in p(x)$.

For any $M \models T$ and any $a \in M^x$, we define $\mathrm{tp}(a) = \{\varphi(x) \mid M \models \varphi(a)\}$. Then $\mathrm{tp}(a)$, the **complete type of** $a$ is a complete type in context $x$.

Just like theories correspond to filters on the Boolean algebra $\mathcal{L}_{()}$ of sentences, and complete theories correspond to ultrafilters on this Boolean algebra, partial types in context $x$ correspond to filters on $\mathcal{L}_x(T)$ and complete types correspond to ultrafilters. The Stone space of ultrafilters on $\mathcal{L}_x(T)$, called the **type space** in context $x$, is denoted $S_x(T)$.

**Proposition 5.1** (Compactness for partial types)**.** *Let $\Sigma(x)$ be a partial type such that every finite subset $\Sigma'(x) \subseteq_{\text{fin}} \Sigma(x)$ is satisfiable (we say $\Sigma(x)$ is **finitely satisfiable**). Then $\Sigma(x)$ is satisfiable.*

*Proof.* If $x = (x_1, \ldots, x_n)$, where $x_i$ has type $s_i$, let $\mathcal{V}' = \mathcal{V} \cup \{c_1, \ldots, c_n\}$, where $c_i$ is a new constant symbol with type $s_i$. Let $T'$ be the $\mathcal{V}'$-theory $T \cup \Sigma(c)$ (where $\Sigma(c)$ is obtained by replacing the variable $x_i$ by $c_i$ in all formulas in $\Sigma(x)$). By hypothesis, every finite subset of $T'$ is satisfiable: For any $\Sigma'(x) \subseteq_{\text{fin}} \Sigma(x)$, if $N \models T$ and $N \models \Sigma'(a)$, then letting $N'$ be the expansion of $N$ in which $c_i^{N'} = a_i$, we have $N' \models T \cup \Sigma'(c)$. By compactness, there exists $M' \models T'$. Letting $a_i = c_i^{M'}$, we have $M' \models \Sigma(a)$. $\square$

A $T$**-definable set** is just a $T$-equivalence class of formulas. Given $T$-definable sets $X$ and $Y$, defined by $\varphi_X(x)$ and $\varphi_Y(y)$ respectively, a $T$**-definable function** $f \colon X \to Y$ is a $T$-equivalence class of formulas $\varphi_f(x, y)$[16] such that

$$T \models \forall x \forall y\, (\varphi_f(x, y) \to (\varphi_X(x) \wedge \varphi_Y(y)))$$
$$T \models \forall x\, (\varphi_X(x) \to \exists^! y\, \varphi_f(x, y))$$

where $\exists^! y\, \varphi(x, y)$ (read "there exists a unique $y$") is shorthand for

$$\exists y\, (\varphi(x, y) \wedge \forall z(\varphi(x, z) \to z = y)).$$

In other words, a definable function is a definable set which is the graph of a function when evaluated in any model of $T$.

[16] Formally, if there is any overlap between the variables $x$ and $y$, we should replace $y$ by a fresh tuple $y'$ with the same types. But let's agree to gloss over this an similar issues.

**Example 5.2.** For any term $t(x)$ of type $s$ in context $x$, the evaluation function $t^M$ is defined by the formula $t(x) = y$. The domain of this function is defined by $\top$ (in context $x$), and the codomain is defined by $\top$ (in context $y$).

But not every definable function is given by a term. For example, let $T$ be the theory of fields. Let $X$ be the set of nonzero elements (defined by the formula $x \neq 0$). Then the multiplicative inverse function $X \to X$ is definable by the formula $x \cdot y = 1$.

Note that we can compose definable functions: If $f\colon X \to Y$ and $g\colon Y \to Z$ are definable functions, then $g \circ f\colon X \to Z$ is definable by $\exists y\,(\varphi_f(x,y) \wedge \varphi_g(y,z))$. And for any definable set $X$ in context $x_1, \ldots, x_n$, we have the identity function $X \to X$, defined by $\varphi_X(x) \wedge \bigwedge_{i=1}^n (x_i = y_i)$.

Now we have some nice categorical structure associated to any theory $T$:

- The category $\mathsf{Def}(T)$, whose objects are $T$-definable sets and whose morphisms are $T$-definable functions.
- The category $\mathsf{Mod}(T)$, whose objects are models of $T$ and whose morphisms are elementary embeddings.
- A functor $\mathrm{Eval} : \mathsf{Def}(T) \times \mathsf{Mod}(T) \to \mathsf{Set}$, whose action on objects is given by $(\varphi(x), M) \mapsto \varphi(M)$.

**Exercise 18.** Define the action of Eval on morphisms, and check that it is a functor.

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We also want to consider definable sets, definable functions, and types **with parameters**. Given a set $A \subseteq M$, recall that $\mathcal{V}(A)$ is the vocabulary obtained from $\mathcal{V}$ by adding a new constant symbol for every element of $A$. And we can view a formula like $\varphi(x; b)$, where $b$ is a tuple from $A$, as an ordinary $\mathcal{V}(A)$-formula. Thus blurring the distinction between constant symbols and parameters, we can simply repeat the definitions above, replacing $T$ by $T(A) = \mathrm{Th}_{\mathcal{V}(A)}(M)$. Note that $T(A)$ is always complete.

- An $A$**-definable set** is just a $T(A)$-definable set.
- An $A$**-definable function** is just a $T(A)$-definable function.
- We write $\mathcal{L}_x(A)$ for the Boolean algebra of $A$-definable sets in context $x$.
- We write $S_x(A)$ for the Stone space of $\mathcal{L}_x(A)$, the **type space over** $A$ in context $x$.
- For any $a \in M^x$, define $\mathrm{tp}(a/A) = \{\varphi(x) \in \mathcal{L}_x(A) \mid M \models \varphi(a)\}$, the **complete type of** $a$ **over** $A$.

**Proposition 5.3** (Realizing types)**.** *Suppose $A \subseteq M$ and $p(x) \in S_x(A)$. Then $p(x)$ is realized in some elementary extension of $M$.*

*Proof.* By definition, $p(x)$ is satisfiable in a model $N$ of $T(A) = \mathrm{Th}_{\mathcal{V}(A)}(M)$, i.e. there is a tuple $b$ in $N^x$ such that $N \models T(A) \cup p(b)$. It suffices to replace $T(A)$ by $\mathrm{EDiag}(M) = \mathrm{Th}_{\mathcal{V}(M)}(M)$.

So in the vocabulary $\mathcal{V}(M)$, consider the partial type $\mathrm{EDiag}(M) \cup p(x)$. We seek to show that this partial type is finitely satisfiable. Since $\mathrm{EDiag}(M)$ and $p(x)$ are both closed under conjunction, it suffices to consider a single formula in $\mathrm{EDiag}(M)$ and a single formula in $p(x)$. We write these as $\psi(a, m)$ and $\varphi(a, x)$, respectively, assuming that $a$ is a tuple enumerating all elements of $A$ that are mentioned in either formula, and $m$ is a tuple enumerating those elements of $M \setminus A$ mentioned in $\psi$.

Now since $M \models \psi(a, m)$, $\exists y\, \psi(a, y) \in T(A)$. So $N \models \exists y\, \psi(a, y)$. Letting $n \in N^y$ be witnesses for the existential quantifiers, and letting $N'$ be the expansion of $N$ to a $\mathcal{V}(M)$-structure by interpreting $m_i^{N'} = n_i$ and the rest of the constants arbitrarily (whenever a constant $c$ has type $s$, $\exists x \top \in T(A)$ when $x$ has type $s$, so $N_s$ is nonempty), we have that $N' \models \psi(a, m) \wedge \varphi(a, b)$. By compactness, $\mathrm{EDiag}(M) \cup p(x)$ is satisfiable, as was to be shown. $\square$

**Example 5.4.** We can now see how classical algebraic geometry is closely related to the model theory of algebraically closed fields. An affine variety $V$ is defined by a set of of polynomials $\{p_1(x_1, \ldots, x_n), \ldots, p_k(x_1, \ldots, x_n)\}$. If the coefficients of the $p_i$ come from a field $k$, we say the variety is defined over $k$. Then for any algebraically closed field extension $F$ of $k$, the set of $F$-points of $V$ is

$$V(F) = \{(a_1, \ldots, a_n) \in F^n \mid p_i(a_1, \ldots, a_n) = 0 \text{ for all } 1 \leq i \leq k\}.$$

Of course, this is just a definable set with parameters from $k$, defined by $\bigwedge_{i=1}^k p_i(x_1, \ldots, x_n) = 0$. The fact that we can consider the $F$-points of $V$ for various fields $F$ corresponds to the ability to evaluate definable sets in various models (as long as they extend the field of definition $k$).

Similarly, a morphism of affine varieties $V \to W$ is given by a polynomial in each coordinate, modulo agreement on all points of $V$. Every such morphism is exactly a definable function between the corresponding definable sets.

We will see later, as a consequence of quantifier elimination for $\mathrm{ACF}_p$, that all embeddings of algebraically closed fields are elementary embeddings, all definable sets are Boolean combinations of zero-sets of polynomials (these are called **constructible sets** in the Zariski topology), and all definable functions between definable sets are rational maps (also allowing $p^{\text{th}}$ roots in the case of characteristic $p$).

## 5.2 Syntactic classes of formulas and equivalence

We now identify some syntactic classes of formulas that we will be particularly interested in. For all $n \in \omega$, we define the classes of $\exists_n$-formulas (also called $\Sigma_n$) and $\forall_n$-formulas (also called $\Pi_n$).

- A **quantifier-free** formula is a formula which contains no quantifiers. We define both the class of $\exists_0$**-formulas** and the class of $\forall_0$**-formulas** to consist of the quantifier-free formulas.

- The $\exists_{n+1}$**-formulas** are the smallest class containing the $\forall_n$-formulas and closed under the formula-building operations $\wedge$, $\vee$, and $\exists$ (no $\neg$ or $\forall$).

- The $\forall_{n+1}$**-formulas** are the smallest class containing the $\exists_n$-formulas and closed under the formula-building operations $\wedge$, $\vee$, and $\forall$ (no $\neg$ or $\exists$).

So the number $n$ counts the quantifier complexity in the sense of the number of alternations of types of quantifiers in building up the formula. We will be particularly interested in the $\exists_n$-formulas and $\forall_n$-formulas when $n \leq 2$: We often call $\exists_1$-formulas simply $\exists$**-formulas** or **existential formulas**, $\forall_1$-formulas simply $\forall$**-formulas** or **universal formulas**, and $\forall_2$-formulas simply $\forall\exists$**-formulas**.

Note that we have $\exists_n \subseteq \forall_{n+1}$ and $\forall_n \subseteq \exists_{n+1}$ for all $n \in \omega$, each class $\exists_n$ and $\forall_n$ contains $\top$ and $\bot$ and is closed under $\wedge$ and $\vee$, and (by induction), the negation of an $\exists_n$-formula is logically equivalent to a $\forall_n$-formula and vice versa.

**Exercise 19.** Show that every quantifier-free formula $\varphi(x)$ is logically equivalent to a formula in **disjunctive normal form**:

$$\bigvee_{i=1}^{n}\left(\bigwedge_{j=1}^{m_i}\chi_{ij}(x)\right),$$

where each $\chi_{ij}(x)$ is atomic or negated atomic. This is really an exercise about Boolean algebra, using de Morgan's laws and distributivity.

Our goal in this section is to understand when a formula or partial type is $T$-equivalent one in a nice syntactic form like those above.

**Lemma 5.5.** *Let $n \geq 1$. Let $p(x)$ and $q(x)$ be complete types, and suppose that every $\exists_n$-formula in $p(x)$ is in $q(x)$. Then if $M \models p(a)$, there exists a structure $N$ and an embedding $h\colon M \to N$ which preserves $\exists_n$-formulas and such that $N \models q(h(a))$.*

*Proof.* Consider the $\mathcal{V}(M)$-theory $\mathrm{Diag}_{\exists_n}(M) \cup q(a)$, where $\mathrm{Diag}_{\exists_n}(M)$ is the set of all $\exists_n$ $\mathcal{V}(M)$-sentences true in $M$. It suffices to show that this theory is consistent.

By compactness, and since $\mathrm{Diag}_{\exists_n}(M)$ is closed under conjunction, it suffices to show that for any formula $\psi(a, m) \in \mathrm{Diag}_{\exists_n}(M)$, the $\mathcal{V}(M)$-theory $\{\psi(a, m)\} \cup q(a)$ is consistent. Here the constants $m$ are those which appear in $\psi$ but not in the tuple $a$.

Since $M \models \exists y\, \psi(a, y)$ and $p(x)$ is complete, $\exists y\, \psi(x, y) \in p(x)$. And this formula is $\exists_n$, so it is in $q(x)$. Then for any realization of $q(x)$, $M' \models q(a')$, there is some $m' \in M'$ such that $M' \models \psi(a', m')$. So $M'$ witnesses that $\{\psi(a, m)\} \cup q(a)$ is consistent, interpreting the constants $a$ as $a'$, $m$ as $m'$, and the remaining constants naming elements of $M$ arbitrarily (here we use the fact that when a sort $M_s$ is nonempty, the $\exists_1$-formula $\exists z\, \top \in p(x)$ where $z$ has type $s$, so it is also in $q(x)$, and it follows that $M'_s$ is nonempty). $\square$

Let's extend some terminology from formulas to partial types (and hence also to theories, which are partial types in the empty context). Partial types $\Sigma(x)$ and $\Sigma'(x)$ are **$T$-equivalent** if for every model $M \models T$ and every $a \in M^x$, $M \models \Sigma(a)$ if and only if $M \models \Sigma'(a)$.

Given a partial type $\Sigma(x)$, we define $[\Sigma(x)] = \bigcap_{\psi(x)\in\Sigma(x)}[\psi(x)] = \{p(x) \in S_x(T) \mid \Sigma(x) \subseteq p(x)\}$. This is a closed set in $S_x(T)$. We also have that $\Sigma(x)$ and $\Sigma'(x)$ are $T$-equivalent if and only if $[\Sigma(x)] = [\Sigma'(x)]$. Indeed, there is a tuple satisfying $\Sigma(x)$ but not $\Sigma'(x)$ if and only if there is a complete type in $[\Sigma(x)]$ but not in $[\Sigma'(x)]$.

Let $p(x)$ and $q(x)$ be complete types. A formula $\varphi(x)$ **separates** $p$ from $q$ if $\varphi(x) \in p(x)$ and $\varphi(x) \notin q(x)$. Note that this notion is not symmetric in $p$ and $q$, but $\varphi(x)$ separates $p$ from $q$ if and only if $\neg\varphi(x)$ separates $q$ from $p$.

**Lemma 5.6** (Separation lemma for partial types)**.** *Suppose $\Delta$ is a class of formulas in context $x$ which contains $\bot$ and is closed under $\vee$, up to $T$-equivalence. Let $\Sigma(x)$ be a partial type, such that for any complete types $p$ and $q$ such that $\Sigma(x) \subseteq p$ and $\Sigma(x) \not\subseteq q$, there is a $\Delta$-formula which separates $p$ from $q$. Then $\Sigma(x)$ is $T$-equivalent to a set of $\Delta$-formulas.*

*Proof.* Fix a type $q \notin [\Sigma(x)]$. For any $p \in [\Sigma(x)]$, there is some $\psi_p(x) \in \Delta$ such that $p \in [\psi_p(x)]$ and $q \notin [\psi_p(x)]$. It follows that $[\Sigma(x)] \subseteq \bigcup_{p\in[\Sigma(x)]}[\psi_p(x)]$. By compactness ($[\Sigma(x)]$ is closed, hence compact), this cover has a finite subcover $[\Sigma(x)] \subseteq \bigcup_{i=1}^n [\psi_{p_i}(x)] = [\bigvee_{i=1}^n \psi_{p_i}(x)]$, and $\bigvee_{i=1}^n \psi_{p_i}(x)$ is $T$-equivalent to some formula $\chi_q(x) \in \Delta$. In the case that the disjunction is empty, $\chi_q(x)$ is $\bot$.

Now we have $[\Sigma(x)] \subseteq [\chi_q(x)]$ and $q \in [\neg\chi_q(x)]$ for all $q \notin [\Sigma(x)]$. So letting $\Sigma'(x) = \{\chi_q(x) \mid q \notin [\Sigma(x)]\}$, we have $[\Sigma(x)] = \bigcap_{q\notin[\Sigma(x)]}[\chi_q(x)] = [\Sigma'(x)]$. $\square$

The proof above used topological compactness in the type space $S_x(T)$. It is also possible to rewrite the proof purely in terms of formulas. For example, the assertion that $[\Sigma(x)] \subseteq \bigcup_{p\in[\varphi(x)]}[\psi_p(x)]$ is equivalent to saying that the partial type $T \cup \Sigma(x) \cup \{\neg\psi_p(x) \mid \varphi(x) \in p(x)\}$ is inconsistent. By compactness, there are finitely many $p_1, \dots, p_n$ such that $T \cup \Sigma(x) \cup \{\neg\psi_{p_i}(x) \mid 1 \leq i \leq n\}$ is inconsistent, which is equivalent to our conclusion that $\Sigma(x) \subseteq \bigcup_{i=1}^n [\psi_{p_i}(x)]$. But topological intuition can be very helpful in discovering proofs like these, i.e. in seeing exactly how to apply compactness.

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**Lemma 5.7.** *Suppose a formula $\varphi(x)$ is $T$-equivalent to a partial type $\Sigma(x)$. Then there exist finitely many formulas $\psi_1(x), \dots, \psi_n(x) \in \Sigma(x)$ such that $\varphi(x)$ is $T$-equivalent to $\bigwedge_{i=1}^n \psi_i(x)$.*

*Proof.* Since $[\varphi(x)] = [\Sigma(x)] = \bigcap_{\psi(x)\in\Sigma(x)}[\psi(x)]$, $[\neg\varphi(x)] = \bigcup_{\psi(x)\in\Sigma(x)}[\neg\psi(x)]$. By compactness, picking a finite subcover, $[\neg\varphi(x)] = \bigcup_{i=1}^n [\neg\psi_i(x)]$, so $[\varphi(x)] = \bigcap_{i=1}^n [\psi_i(x)] = [\bigwedge_{i=1}^n \psi_i(x)]$. $\square$

**Corollary 5.8** (Separation lemma for formulas)**.** *Suppose $\Delta$ is a class of formulas in context $x$ which contains $\top$ and $\bot$ and is closed under $\wedge$ and $\vee$, up to $T$-equivalence. Let $\varphi(x)$ be a formula, such that for any complete types $p$ and $q$ such that $\varphi(x) \in p$ and $\varphi(x) \notin q$, there is a $\Delta$-formula which separates $p$ from $q$. Then $\varphi(x)$ is $T$-equivalent to a $\Delta$-formula.*

*Proof.* Applying Lemma 5.6 to the partial type $\{\varphi(x)\}$, we find that $\varphi(x)$ is $T$-equivalent to a set $\Sigma(x)$ of $\Delta$-formulas. By Lemma 5.7, $\varphi(x)$ is $T$-equivalent to $\bigwedge_{i=1}^{n} \psi_i(x)$, where each $\psi_i(x) \in \Sigma(x)$. Since $\Delta$ contains $\top$ and is closed under $\wedge$ up to $T$-equivalence, $\varphi(x)$ is $T$-equivalent to a $\Delta$-formula. $\square$

## 5.3 Up and down: $\exists$-formulas and $\forall$-formulas

**Proposition 5.9.** *Embeddings preserve $\exists$-formulas.*

*Proof.* Let $h\colon M \to N$ be an embedding. We argue that $h$ preserves $\varphi(x)$ by induction on the complexity of the construction of $\varphi(x)$ as an $\exists$-formula.

In the base case, $h$ preserves quantifier-free formulas, and the induction steps for $\wedge$ and $\vee$ are easy. So it suffices to consider the case when $\varphi(x)$ is $\exists y\, \psi(x, y)$, and by induction $h$ preserves $\psi(x, y)$. If $M \models \varphi(a)$, then there is some $b \in M^y$ such that $M \models \psi(a, b)$. Then $N \models \psi(h(a), h(b))$, so $N \models \varphi(h(a))$. $\square$

A partial type $\Sigma(x)$ is a partial $\exists_n$-type when every formula in it is a $\exists_n$-formula. We use the same terminology for partial $\forall_n$-types, $\exists_n$-theories, and $\forall_n$-theories.

**Theorem 5.10.** *A partial type $\Sigma(x)$ is preserved by embeddings between models of $T$ if and only if it is $T$-equivalent to a partial $\exists$-type.*

*Proof.* One direction is immediate from Proposition 5.9. For the converse, we appeal to Lemma 5.6. Let $p(x)$ and $q(x)$ be complete types such that $\Sigma(x) \subseteq p(x)$ and $\Sigma(x) \not\subseteq q(x)$. Suppose for contradiction that there is no $\exists$-formula separating $p(x)$ from $q(x)$. Then every $\exists$-formula in $p(x)$ is in $q(x)$. By Lemma 5.5, picking a realization $M \models p(a)$, there is an embedding $h\colon M \to N$ such that $N \models q(h(a))$.

But then $M \models \Sigma(a)$, and $\Sigma(x)$ is preserved by embeddings, so $N \models \Sigma(h(a))$. Since also $N \models q(h(a))$ and $q(x)$ is complete, $\Sigma(x) \subseteq q(x)$. This is a contradiction. $\square$

**Theorem 5.11.** *A partial type $\Sigma(x)$ is reflected by embeddings between models of $T$ if and only if it is $T$-equivalent to a partial $\forall$-type.*

*Proof.* Proposition 5.9 implies that embeddings reflect $\forall$-formulas, since these are exactly the negations of $\exists$-formulas. So again, one direction of the theorem is immediate.

For the converse, we again appeal to Lemma 5.6. Let $p(x)$ and $q(x)$ be complete types such that $\Sigma(x) \subseteq p(x)$ and $\Sigma(x) \not\subseteq q(x)$. Suppose for contradiction that there is no $\forall$-formula separating $p(x)$ from $q(x)$. Then every $\forall$-formula in

$p(x)$ is in $q(x)$. Turning this around, every $\exists$-formula in $q(x)$ is in $p(x)$, since $p$ and $q$ are complete. By Lemma 5.5, picking a realization $M \models q(a)$, there is an embedding $h\colon M \to N$ such that $N \models p(h(a))$.

But then $N \models \Sigma(h(a))$, and $\Sigma(x)$ is reflected by embeddings, so $M \models \Sigma(a)$. Since also $N \models q(a)$ and $q(x)$ is complete, $\Sigma(x) \subseteq q(x)$. This is a contradiction. $\square$

It follows immediately from Lemma 5.7 that a formula is preserved (respectively, reflected) by embeddings between models of $T$ if and only if it is $T$-equivalent to an $\exists$-formula (respectively, a $\forall$-formula).

Here are some more variants of the results above.

**Exercise 20.** Let $T$ be any theory.

(a) The class of models of $T$ is closed under substructure if and only if $T$ is equivalent to a $\forall$ theory.

(b) Let $T_\forall$ be the set of universal consequences of $T$. Then the class of models of $T_\forall$ is exactly the class of substructures of models of $T$.

As an example of Exercise 20(a), the theory of groups in the vocabulary $(\cdot, e, ^{-1})$ is axiomatizable by universal sentences, and indeed the class of groups is closed under substructure. It is also possible to axiomatize groups in the vocabulary $(\cdot)$, but this time universal sentences do not suffice, e.g. we need an axiom like $\exists x\,\forall y\,(x \cdot y = y \wedge y \cdot x = y)$. And indeed, in this vocabulary there are substructure of groups which are not subgroups, e.g. $(\mathbb{N}, +)$ is a substructure of $(\mathbb{Z}, +)$.

As an example of Exercise 20(b), Letting $T = \mathrm{ACF}$, $T_\forall$ is equivalent to the theory of integral domains.

**Exercise 21.** A formula is **positive quantifier-free** if it is built up from atomic formulas by $\top$, $\bot$, $\wedge$, and $\vee$ (no quantifiers or negation). A $\exists^+$**-formula**, also called a **positive existential formula**, is built up from positive quantifier-free formulas by $\wedge$, $\vee$, and $\exists$. Show that a formula $\varphi(x)$ is preserved by homomorphisms between models of $T$ if and only if it is $T$-equivalent to a $\exists^+$-formula. Which formulas are reflected by homomorphisms between models of $T$?

## 5.4 Directed colimits: $\forall\exists$ formulas

A poset $(I, \leq)$ is **directed** if it is nonempty, and for all $i, j \in I$ there exists $k \in I$ such that $i \leq k$ and $j \leq k$. A **directed system**, indexed by the directed poset $(I, \leq)$, is a family of $\mathcal{V}$-structures $(M_i)_{i \in I}$ and a family of homomorphisms $f_{ij}\colon M_i \to M_j$ for all $i \leq j$ in $I$, such that $f_{ii}$ is the identity map on $M_i$ for all $i \in I$, and if $i \leq j$ and $j \leq k$, then $f_{jk} \circ f_{ij} = f_{ik}$. That is, it is (the image of) a functor from the poset $(I, \leq)$ (viewed as a category) to the category of $\mathcal{V}$-structures and homomorphisms.

A **chain** is a directed system indexed by the poset $(\omega, \leq)$:

$$M_0 \to M_1 \to M_2 \to \dots$$

Given a directed system $(M_i)_{i\in I}$ (we usually omit the $f_{ij}$ from the notation), we define the **directed colimit** $M = \varinjlim M_i$.

The domain is the quotient of the disjoint union $\bigsqcup_{i\in I} M_i$ by the following equivalence relation: If $a \in M_i$ and $b \in M_j$, then $a \sim b$ if and only if there is some $i \leq k$ and $j \leq k$ such that $f_{ik}(a) = f_{jk}(b)$. Directedness of $(I, \leq)$ is used in checking transitivity of this relation. For each $i \in I$, there is a map $\sigma_i \colon M_i \to M$, sending $a$ to its equivalence class.

For each function symbol $f \in \mathcal{F}$ and each possible input tuple $b = (b_1, \ldots, b_n)$ from $M$, by directedness there is some $i \in I$ such that $b$ is in the image of $\sigma_i$, so $b_j = \sigma_i(a_j)$ for all $1 \leq j \leq n$. Define $f^M(b_1, \ldots, b_n) = \sigma_i(f^{M_i}(a_1, \ldots, a_n))$.

Similarly, for each relation symbol $R \in \mathcal{R}$ and each possible tuple $b = (b_1, \ldots, b_n)$ from $M$, set $(b_1, \ldots, b_n) \in R^M$ if and only if there exists $i \in I$ and $(a_1, \ldots, a_n) \in R^{M_i}$ such that $b_j = \sigma_i(a_j)$ for all $1 \leq j \leq n$.

**Exercise 22.** Check that:

(a) $\varinjlim M_i$ is well-defined as a $\mathcal{V}$-structure.

(b) Each map $\sigma_i$ is a homomorphism.

(c) If each map $f_{ij}$ is an embedding, then each map $\sigma_i$ is an embedding.

(d) $\varinjlim M_i$ satisfies the universal property of the colimit: Given a structure $N$ and a homomorphism $h_i \colon M_i \to N$ for all $i \in I$, such that $h_j \circ f_{ij} = h_i$ whenever $i \leq j$, there is a unique homomorphism $h \colon \varinjlim M_i \to N$ such that $h \circ \sigma_i = h_i$ for all $i \in I$. And if all the maps $f_{ij}$ and $h_i$ are embeddings, then $h$ is an embedding. That is, our definition of $\varinjlim M_i$ gives the colimit in the category of $\mathcal{V}$-structures and homomorphisms, and in the category of $\mathcal{V}$-structures and embeddings.

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A directed colimit along a chain is called a **chain colimit**. If each map $f_{ij}$ is an (elementary) embedding, we say that $\varinjlim M_i$ is a **directed colimit along (elementary) embeddings**. The directed colimit of a directed system $(M_i)_{i\in I}$ along embeddings can essentially be thought of as $\bigcup_{i\in I} M_i$, if we think of each embedding $f_{ij}$ as an inclusion.

**Example 5.12.** Every $\mathcal{V}$-structure $M$ is the directed colimit of its finitely generated substrutures. Here the directed system has objects $\{\langle A\rangle \mid A \subseteq_{\text{fin}} M\}$ and embeddings all the inclusions $\langle A\rangle \subseteq \langle B\rangle$.

**Proposition 5.13.** *If $M = \varinjlim M_i$ is a directed colimit along elementary embeddings, then each map $\sigma_i$ is an elementary embedding.*

*Proof.* Let $\varphi(x)$ be a formula. We show by induction on the complexity of $\varphi(x)$ that for all $i \in I$ and all $a \in M_i^x$, $M_i \models \varphi(a)$ if and only if $M \models \varphi(\sigma_i(a))$.

By Exercise 22, each map $\sigma_i \colon M_i \to M$ is an embedding, so it preserves and reflects atomic formulas. This handles the base case. The inductive steps for Boolean combinations are easy.

So we consider the case when $\varphi(x)$ is $\exists y\, \psi(x,y)$. Suppose $M_i \models \exists\varphi(a)$. Letting $b$ witness the quantifier, $M_i \models \psi(a,b)$, so $M \models \psi(\sigma_i(a), \sigma_i(b))$ by induction, and $M \models \varphi(\sigma_i(a))$.

Conversely, suppose $M \models \varphi(\sigma_i(a))$. Letting $b$ witness the quantifier, there is some $j \in I$ and some $b' \in M_j$ such that $b = \sigma_j(b')$. By directedness, we may assume that $j \geq i$. Then $M \models \psi(\sigma_j(f_{ij}(a)), \sigma_j(b'))$, and by induction $M_j \models \psi(f_{ij}(a), b')$, so $M_j \models \varphi(f_{ij}(a))$. But $f_{ij}$ is an elementary embedding, so $M_i \models \varphi(a)$. $\square$

We say that a formula $\varphi(x)$ is **preserved in the directed colimit** $M = \varinjlim M_i$ if for all $i \in I$, and all $a \in M_i^x$, if $M_j \models \varphi(f_{ij}(a))$ for all $j \geq i$, then $M \models \varphi(\sigma_i(a))$.

**Proposition 5.14.** *Directed colimits along embeddings preserve $\forall\exists$-formulas.*

*Proof.* Let $\varphi(x)$ be a $\forall\exists$-formula, and let $M = \varinjlim M_i$ be a directed colimit along embeddings. We show by induction on the construction of $\varphi(x)$ as a $\forall\exists$-formula that it is preserved in the directed colimit.

In the base case, when $\varphi(x)$ is an $\exists$-formula, if $M_i \models \varphi(a)$, then $M \models \varphi(\sigma_i(a))$ by Proposition 5.9, since $\sigma_i$ is an embedding.

The inductive steps for $\wedge$ and $\vee$ are easy. So we consider the case when $\varphi(x)$ is $\forall y\, \psi(x,y)$. We assume that $M_j \models \varphi(f_{ij}(a))$ for all $j \geq i$. To show that $M \models \varphi(\sigma_i(a))$, let $b \in M^y$. Then there is some $j \in I$ and some $b' \in M_j$ such that $b = \sigma_j(b')$. By directedness, we may assume that $j \geq i$. Now for all $k \geq j$, $M_k \models \forall y\, \psi(f_{ik}(a), y)$, so $M_k \models \psi(f_{jk}(f_{ij}(a)), f_{jk}(b'))$. By induction, $M \models \psi(\sigma_j(f_{ij}(a)), \sigma_j(b'))$, so $M \models \psi(\sigma_i(a), b)$. Since $b$ was arbitrary, $M \models \varphi(\sigma_i(a))$, as desired. $\square$

**Theorem 5.15.** *Let $\Sigma(x)$ be a partial type. The following are equivalent:*

*(1) $\Sigma(x)$ is preserved by directed colimits of models of $T$ along embeddings.*

*(2) $\Sigma(x)$ is preserved by chain colimits of models of $T$ along of embeddings.*

*(3) $\Sigma(x)$ is $T$-equivalent to a partial $\forall\exists$-type.*

*Proof.* (3)$\Rightarrow$(1) is immediate from Proposition 5.14.

(1)$\Rightarrow$(2) is trivial, since chain colimits are special kinds of directed colimits.

(3)$\Rightarrow$(1): We use Lemma 5.6. Let $p(x)$ and $q(x)$ be complete types such that $\Sigma(x) \subseteq p(x)$ and $\Sigma(x) \not\subseteq q(x)$. Suppose for contradiction that there is no $\forall_2$-formula separating $p(x)$ from $q(x)$. Then every $\forall_2$-formula in $p(x)$ is in $q(x)$. Turning this around, every $\exists_2$-formula in $q(x)$ is in $p(x)$.

Pick an arbitrary realization of $q(x)$: Let $M_0 \models T$ and $a \in M_0^x$ such that $M_0 \models q(a)$. By Lemma 5.5, there exists a structure $N_0$ and an embedding $h_0\colon M_0 \to N_0$ which preserves $\exists_2$-formulas and such that $N_0 \models p(h_0(a))$. In particular, $h$ preserves $\forall_1$-formulas and reflects $\exists_1$-formulas.

Now view $N_0$ as a $\mathcal{V}(M_0)$-structure by interpreting each constant symbol $m$ as $h_0(M)$. Then $\mathrm{Th}_{\mathcal{V}(M_0)}(N_0)$ and $\mathrm{EDiag}(M_0)$ are complete theories (complete types in the empty context), and every $\exists_1$-sentence in $\mathrm{Th}_{\mathcal{V}(M_0)}(N_0)$ is in

$\mathrm{EDiag}(M_0)$, since $h$ reflects $\exists_1$-formulas. By Lemma 5.5 again, there is a model $M_1 \models \mathrm{EDiag}(M_0)$ and an embedding $h_0'\colon N_0 \to M_1$.

Since $M_1 \models \mathrm{EDiag}(M_0)$, the interpretation of the constants defines an elementary embedding $M_0 \to M_1$ (we assume for simplicity that $M_1$ is an elementary extension of $M_0$). Since $h_0'$ is an $\mathcal{V}(M_0)$-embedding, the diagram commutes:

$$\begin{tikzcd}
 & N_0 \arrow[dr, "h_0'"] & \\
M_0 \arrow[ur, "h_0"] \arrow[rr, "\preceq"'] & & M_1
\end{tikzcd}$$

Since $M_0 \preceq M_1$, $M_1 \models q(a)$, and we can repeat the construction. By induction, we obtain the following commutative diagram:

$$\begin{tikzcd}
 & N_0 \arrow[rr, "h_1 \circ h_0'"] \arrow[dr, "h_0'"] & & N_1 \arrow[rr, "h_2 \circ h_1'"] \arrow[dr, "h_1'"] & & \cdots & & N_\omega \arrow[d, leftrightarrow] \\
M_0 \arrow[ur, "h_0"] \arrow[rr, "\preceq"'] & & M_1 \arrow[ur, "h_1"] \arrow[rr, "\preceq"'] & & M_2 \arrow[ur, "h_2"] \arrow[r] & \cdots & & M_\omega
\end{tikzcd}$$

Let $M_\omega = \varinjlim_{i\in\omega} M_i$ and $N_\omega = \varinjlim_{i\in\omega} N_i$. The map $h\colon M_\omega \to N_\omega$ induced by the $h_i$ and the map $h'\colon N_\omega \to M_\omega$ induced by the $h_i'$ are inverses, so $M_\omega \cong N_\omega$.

Since $M_\omega$ is a directed colimit along elementary embeddings, $M_0 \preceq M_\omega$. In particular, $M_\omega \models T$ (so also $N_\omega \models T$), and $M_\omega \models q(a)$.

Since $N_0 \models \Sigma(h_0(a))$ and $\Sigma(x)$ is preserved by chain colimits of models of $T$ along embeddings, $N_\omega \models \Sigma(h(a))$. But $h$ is an isomorphism, so also $M_\omega \models \Sigma(a)$, and since $q(x)$ is complete, $\Sigma(x) \subseteq q(x)$. This is a contradiction. $\square$

Just as in Exercise 20, the class of models of a theory $T$ is closed under directed colimits if and only if $T$ is equivalent to a $\forall\exists$-theory. And it follows immediately from Lemma 5.7 that a formula is preserved by directed colimits of models of $T$ along embeddings if and only if it is $T$-equivalent to a $\forall\exists$-formula.

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We conclude this section with a nice application (which only uses the "easy direction" Proposition 5.14, not the full strength of Theorem 5.15).

**Theorem 5.16** (Ax–Grothendieck). *Any injective polynomial map $\mathbb{C}^n \to \mathbb{C}^n$ is surjective.*

In fact, we will prove a stronger theorem and derive the Ax–Grothendieck theorem as a corollary.

**Theorem 5.17.** *Let $\varphi$ be a $\forall\exists$-sentence in the language of rings which is true in every finite field. Then $\varphi$ is true in every algebraically closed field.*

*Proof.* Fix a prime number $p$, and let $K = \overline{\mathbb{F}_p}$, the algebraic closure of the prime field $\mathbb{F}_p$. Then $K$ is the directed colimit of its finitely generated substructures, which are all finite fields, using the fact that a finitely generated algebraic extension of $\mathbb{F}_p$ is finite.

More explicitly, $K = \varinjlim(\mathbb{F}_{p^n})_{n>0}$, where the index set $\{n \in \omega \mid n > 0\}$ is not ordered by the usual order $\leq$, but rather by the divisibility order $|$.

Now we assumed that $\varphi$ is true in every finite field, and since it is $\forall\exists$, it is preserved in directed colimits, so $K \models \varphi$, and by completeness, $\mathrm{ACF}_p \models \varphi$.

We have shown that $\varphi$ holds in all algebraically closed fields of finite characteristic. It follows that it holds in all algebraically closed fields of characteristic 0 by the transfer theorem, Corollary 4.20. $\square$

*Proof of Theorem 5.16.* To say that $f\colon \mathbb{C}^n \to \mathbb{C}^n$ is a polynomial map is to say that $f$ is given by $n$ polynomials $(p_i)_{i=1}^n$, with each $p_i \in \mathbb{C}[x_1, \dots, x_n]$, and $f(a_1, \dots, a_n) = (p_1(a_1, \dots, a_n), \dots, p_n(a_1, \dots, a_n))$.

Now for each degree $d$, we can express the statement of the theorem, restricted to polynomials of degree at most $d$, by the following sentence $\varphi_d$:

$$\forall f (\forall x\, \forall x' ((f(x) = f(x')) \to (x = x')) \to \forall y\, \exists x\, (f(x) = y))$$

There are a few things to explain here.

(1) The quantifier $\forall f$ means to quantify over all systems of $n$ polynomials in $n$ variables of degree at most $d$, which means to quantify over the coefficients of these polynomials. So really we have $n(d+1)$ quantifiers, over variables $(a_i^j)_{1\leq i\leq n, 0\leq j\leq d}$, which specify polynomials $p_i = a_i^0 + a_i^1 x + \cdots + a_i^d x^d$ for $1 \leq i \leq n$.

(2) Similarly, the quantifiers over $x$, $x'$ and $y$ are over $n$-tuples, e.g. $x = (x_i)_{1\leq i\leq n}$.

(3) The formulas $f(x) = f(x')$, $x = x'$, and $f(x) = y$ are really conjunctions of $n$ atomic formulas, e.g. $f(x) = f(x')$ is $\bigwedge_{i=1}^n (p_i(x) = p_i(x'))$.

(4) $\varphi_d$ is logically equivalent to a $\forall\exists$-sentence. We have to be slightly careful about the presence of $\to$, because $\varphi \to \psi$ is shorthand for $\neg\varphi \vee \psi$, and $\neg$ is not allowed in the construction of $\forall\exists$-formulas. But we can rewrite $\varphi_d$ as the logically equivalent $\forall\exists$-sentence:

$$\forall f\, (\exists x\, \exists x'\, \neg((f(x) = f(x')) \to (x = x')) \vee \forall y\, \exists x\, (f(x) = y)).$$

Now for all $d$ and any finite field $F$, we have $F \models \varphi_d$, since an injective map from a finite set to itself is always surjective. By Theorem 5.17, $\mathbb{C} \models \varphi_d$ for all $d$, as desired. $\square$

**Exercise 23.** (a) The converse of the Ax–Grothendieck theorem fails: $x \mapsto x^2$ is a surjective but not injective polynomial map $\mathbb{C} \to \mathbb{C}$. Where exactly does the proof go wrong?

(b) Prove the following strengthening of Ax–Grothendieck: Let $V \subseteq \mathbb{C}^n$ be an algebraic set (an algebraic set is the set of zeros of a finite family of polynomials, i.e. $V = \{a \in \mathbb{C}^n \mid q_i(a) = 0 \text{ for all } 1 \leq i \leq k\}$, where $q_i \in \mathbb{C}[x_1, \dots, x_n]$ for all $1 \leq i \leq k$). Then any injective polynomial map $V \to V$ is surjective.

# 6 Quantifier elimination

## 6.1 Fraïssé limits

Tuesday 10/23 For simplicity, we will restrict our attention to single-sorted finite relational vocabularies $\mathcal{V}$ (relational means there are no function symbols or constant symbols). Most of Fraïssé theory works without these restrictions, and in even more general contexts. But I think it's best at the beginning to avoid the technicalities that arise.

These restrictions have two main consequences for us. First, any subset of a $\mathcal{V}$-structure is the domain of an induced substructure (there's no need to close under the interpretations of function symbols). Second, for $n \in \omega$, there are only finitely many $\mathcal{V}$-structures of size $n$ up to isomorphism, and there are only countably many finite $\mathcal{V}$-structures up to isomorphism.

Given a structure $M$, we denote by $\mathrm{age}(M)$ the class of all finite structures which embed in $M$ (equivalently, which are isomorphic to substructures of $M$). This terminology and extensions of it were introduced by Fraïssé. For example, Fraïssé says that $M$ is younger than $N$ if $\mathrm{age}(M) \subseteq \mathrm{age}(N)$.

Which classes $\mathcal{K}$ of finite structures occur as the age of some structure? Fraïssé identified the following necessary and sufficient conditions:

- (Isomorphism Closed) If $A \in \mathcal{K}$ and $B \cong A$, then $B \in \mathcal{K}$.
- (Hereditary Property - HP) If $A \in \mathcal{K}$ and $B$ is a substructure of $A$, then $B \in \mathcal{K}$.
- (Joint Embedding Property - JEP) For all $A, B \in \mathcal{K}$, there exists $C \in \mathcal{K}$ and embeddings $f\colon A \to C$ and $g\colon B \to C$.

$$\begin{tikzcd} & C & \\ A \arrow[ur, dashed, "f"] & & B \arrow[ul, dashed, "g"'] \end{tikzcd}$$

**Proposition 6.1.** *For a class $\mathcal{K}$ of finite structures, the following are equivalent:*

*(1) $\mathcal{K} = \mathrm{age}(M)$ for some structure $M$.*

*(2) $\mathcal{K} = \mathrm{age}(M)$ for some countable structure $M$.*

*(3) $\mathcal{K}$ is isomorphism closed and has HP and JEP.*

*Proof.* (2)$\Rightarrow$(1) is trivial.

(1)$\Rightarrow$(3): Let $M$ be any structure. The fact that $\mathrm{age}(M)$ is isomorphism closed and has HP is immediate from the definition: If $f\colon A \to M$ is an embedding, and $B$ is isomorphic to $A$ or a substructure of $A$, then composing with $f$ gives an embedding $B \to M$. For the JEP, suppose $F\colon A \to M$ and $G\colon A \to M$ are embeddings. Let $C$ be the substructure of $M$ with domain

$F(A) \cup G(B)$. Then $C \in \text{age}(M)$ and $F$ and $G$ restrict to embeddings $f: A \to C$ and $g: B \to C$.

(3)⇒(2): Let $\mathcal{K}$ be a class of finite structures which is isomorphism closed and has HP and JEP. Let $(A_i)_{i\in\omega}$ be an enumeration of representatives for the isomorphism classes in $\mathcal{K}$ (there are countably many, since we are in a finite relational vocabulary). We define a chain $(B_i)_{i\in\omega}$ of structures in $\mathcal{K}$ by induction. Let $B_0 = A_0$. Given $B_i$, use JEP to find a structure $B_{i+1}$ with embeddings $f_i: B_i \to B_{i+1}$ and $g_i: A_{i+1} \to B_{i+1}$.

$$\begin{tikzcd}
A_0 = B_0 \arrow[r, "f_0"] & B_1 \arrow[r, "f_1"] & B_2 \arrow[r, "f_2"] & B_3 \arrow[r, "f_3"] & \dots \\
A_1 \arrow[ur, "g_0"'] & A_2 \arrow[ur, "g_1"'] & A_3 \arrow[ur, "g_2"'] & \dots &
\end{tikzcd}$$

Let $M = \varinjlim B_i$. Then $\mathcal{K} \subseteq \text{age}(M)$, since every structure $A$ in $\mathcal{K}$ is isomorphic to $A_i$ for some $i \in \omega$, and $A_i$ embeds in $B_i$, which embeds in $M$. Conversely, suppose $C \in \text{age}(M)$, witnessed by $h: C \to M$. Since $C$ is finite, $h(C)$ is contained in $\sigma_i(B_i)$ for some $i \in \omega$ (where $\sigma_i: B_i \to M$ is the canonical embedding), so $C$ is isomorphic to a substructure of $B_i$, and hence is in $\mathcal{K}$ by HP and isomorphism closure. □

Of course, there may be many countable structures with the same age. For example, any two countable linear orders have the same age, since there is a unique (up to isomorphism) finite linear order of size $n$ for each $n \in \omega$.

Given a class of finite structures $\mathcal{K}$, we say that a structure $M$ is a **Fraïssé limit of** $\mathcal{K}$ if $M$ if countable, $\text{age}(M) = \mathcal{K}$, and $M$ is **$\mathcal{K}$-homogeneous**: If $f: A \to M$ and $g: A \to B$ are embeddings, with $A, B \in \mathcal{K}$, then there exists an embedding $h: B \to M$ such that $f = h \circ g$.

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "g"'] & & M \\
& B \arrow[ur, dashed, "h"'] &
\end{tikzcd}$$

Before tackling the question of the existence of the Fraïssé limit $M$ of $\mathcal{K}$, we will prove that it has the following pleasant properties:

1. It is unique up to isomorphism.

2. If $N$ is any countable structure with $\text{age}(N) \subseteq \mathcal{K}$, then $N$ embeds in $M$.

3. It is **strongly $\mathcal{K}$-homogeneous**: Every isomorphism between finite substructures of $M$ in $\mathcal{K}$ extends to an automorphism of $M$.

Let's pause to note that when $\text{age}(M) = \mathcal{K}$, strong $\mathcal{K}$-homogeneity implies $\mathcal{K}$-homogeneity. If $f: A \to M$ and $g: A \to B$ are embeddings with $A, B \in \mathcal{K}$, then there is another embedding $h': B \to M$, since $B \in \text{age}(M)$. Now there is an isomorphism $\sigma\colon h'(g(A)) \to f(A)$ such that $(\sigma \circ h' \circ g)(a) = f(a)$ for all $a \in A$, and by strong $\mathcal{K}$-homogeneity, this isomorphism extends to an automorphism $\widehat{\sigma}\colon M \to M$. Then $h = \widehat{\sigma} \circ h'$ is an embedding $B \to M$ such that $h \circ g = f$. Indeed, for all $a \in A$, $(h \circ g)(a) = (\widehat{\sigma} \circ h' \circ g)(a) = (\sigma \circ h' \circ g)(a) = f(a)$.

To warm up, we prove point 2 by going "forth".

**Proposition 6.2.** *Suppose $M$ is a Fraïssé limit of $\mathcal{K}$. If $N$ is any countable structure with* $\mathrm{age}(N) \subseteq \mathcal{K}$*, then $N$ embeds in $M$.*

*Proof.* Enumerate $N$ as $(n_i)_{i \in \omega}$, and let $N_i$ be the induced substructure of $N$ with domain $\{n_j \mid j \leq i\}$. Since $\mathrm{age}(N) \subseteq \mathrm{age}(M) = \mathcal{K}$, there is an embedding $f_0\colon N_0 \to M$, and given an embedding $f_i\colon N_i \to M$, we can extend it to an embedding $f_{i+1}\colon N_{i+1} \to M$ by $\mathcal{K}$-homogeneity. The union of the embeddings $(f_i)_{i \in \omega}$ is an embedding $N \to M$. $\square$

Now we prove points 1 and 3 by going "back-and-forth".

**Lemma 6.3.** *Suppose $M$ and $M'$ are Fraïssé limits of $\mathcal{K}$, $A \subseteq M$ and $A' \subseteq M'$ are finite substructures, and $f\colon A \to A'$ is an isomorphism. Then $f$ extends to an isomorphism $\widehat{f}\colon M \to M'$.*

$$\begin{tikzcd}
M \arrow[r, dashed, "\widehat{f}"] & M' \\
A \arrow[u] \arrow[r, "f"] & A' \arrow[u]
\end{tikzcd}$$

*Proof.* Enumerate $M$ as $(m_i)_{i \in \omega}$ and $M'$ as $(m'_i)_{i \in \omega}$. Let $M_0 = A$, $M'_0 = A'$, and $f_0 = f$. We will build a sequence of isomorphisms $f_i\colon M_i \to M'_i$ such that $f_i$ extends $f_j$ when $j \leq i$, $M_i$ contains $\{m_j \mid j < i\}$, and $M'_i$ contains $\{m'_j \mid j < i\}$.

Given $f_i\colon M_i \to M'_i$, let $N_{i+1}$ be the induced substructure of $M$ with domain $M_i \cup \{m_i\}$. Viewing $f_i$ as an embedding $M_i \to M'$, use $\mathcal{K}$-homogeneity to extend it to an embedding $N_{i+1} \to M'$. Let $N'_{i+1}$ be the image of this embedding, and let $g_{i+1}$ be the isomorphism $N_{i+1} \to N'_{i+1}$.

Now let $M'_{i+1}$ be the induced substructure of $M'$ with domain $N'_{i+1} \cup \{m'_i\}$. Viewing $g_{i+1}^{-1}$ as an embedding $N_{i+1}$ to $M$, use $\mathcal{K}$-homogeneity to extend it to an embedding $M'_{i+1} \to M$. Let $M_{i+1}$ be the image of this embedding, and let $f_{i+1}\colon M_{i+1} \to M'_{i+1}$ be the induced isomorphism.

Then the union of the isomorphisms $(f_i)_{i \in \omega}$ is an isomorphism $M \to M'$ extending $f = f_0$, as desired. $\square$

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**Corollary 6.4.** *If $M$ is a Fraïssé limit of $\mathcal{K}$, then $M$ is strongly $\mathcal{K}$-homogeneous.*

*Proof.* This is just Lemma 6.3, taking $M = M'$ and $f\colon A \to A'$ to be the isomorphism between finite substructures. $\square$

**Corollary 6.5.** *$\mathcal{K}$ has at most one Fraïssé limit up to isomorphism.*

*Proof.* Suppose $M$ and $M'$ are Fraïssé limits of $\mathcal{K}$. Again, we apply Lemma 6.3, this time taking $f\colon A \to A'$ to be the unique isomorphism between the empty substructures of $M$ and $M'$.

There is a slight point to be made here: since we allow 0-ary relation symbols, i.e. proposition symbols, in our vocabulary, it is possible to have multiple non-isomorphic $\mathcal{V}$-structures. But a structure $M$ has a unique empty structure, so $\mathrm{age}(M)$ contains a unique empty structure up to isomorphism (this also follows directly from JEP). It follows that the empty substructures of $M$ and $M'$ are isomorphic. □

Thanks to this result, we typically refer to *the* Fraïssé limit of a class $\mathcal{K}$.

We now turn to existence. For $\mathcal{K}$ to admit a Fraïssé limit, it must be at least isomorphism closed with HP and JEP. We need one more necessary and sufficient condition:

- (Amalgamation Property - AP) For all $A, B, C$ in $\mathcal{K}$ and embeddings $f\colon A \to B$ and $g\colon A \to C$, there exists $D \in \mathcal{K}$ and embeddings $f'\colon B \to D$ and $g'\colon C \to D$ such that $f' \circ f = g' \circ g$.

$$\begin{tikzcd}
 & D & \\
B \arrow[ur, dashed, "f'"] & & C \arrow[ul, dashed, "g'"'] \\
 & A \arrow[ul, "f"] \arrow[ur, "g"'] &
\end{tikzcd}$$

$\mathcal{K}$ is a **Fraïssé class** if it is isomorphism closed and has HP, JEP, and AP.

**Theorem 6.6.** *For a class $\mathcal{K}$ of finite structures, the following are equivalent:*

*(1) $\mathcal{K}$ has a Fraïssé limit.*

*(2) $\mathcal{K}$ is a Fraïssé class.*

*Proof.* Suppose first that $\mathcal{K}$ has a Fraïssé limit $M$. Then $\mathcal{K} = \mathrm{age}(M)$, so it is isomorphism closed and has HP and JEP. For AP, let $f\colon A \to B$ and $g\colon A \to C$ be embeddings between structures in $\mathcal{K}$. Since $B \in \mathrm{age}(M)$, there is an embedding $f'\colon B \to M$, and $f' \circ f\colon A \to M$ is also an embedding. Using $\mathcal{K}$-homogeneity, there is an embedding $g'\colon C \to M$ such that $g' \circ g = f' \circ f$. We conclude by taking $D$ to be the substructure of $M$ with domain $f'(B) \cup g'(C)$ and restricting the codomains of $f'$ and $g'$ to $D$.

Conversely, suppose $\mathcal{K}$ is a Fraïssé class. We will build a sequence $(B_i)$ of structures in $\mathcal{K}$ and embeddings $f_i\colon B_i \to B_{i+1}$ by induction, modifying the construction in Proposition 6.1 to ensure $\mathcal{K}$-homogeneity.

Let $(A_i)_{i\in\omega}$ be an enumeration of representatives for the isomorphism classes in $\mathcal{K}$. A **task** is either a structure $A_i$ on the list or an embedding $g\colon C \to A_i$, where $A_i$ is some structure on the list and $C$ is a substructure of some $B_i$ in our inductive construction. We will list the tasks by pairs in $\omega \times \omega$. To start, define task $(i, 0)$ to be the structure $A_i$. We will define tasks $(i, j)$ for $j > 0$ during stage $i$ of the construction. Fix a bijection[17] $t\colon \omega \to \omega \times \omega$ such that $t(0) = (0, 0)$ and $t(n) = (m, m')$ with $m < n$ for all $n > 0$ (this ensures that task $t(n)$ has already been defined before stage $n$). We will accomplish task $t(n)$ at stage $n$ of the construction.

At stage 0, let $B_0 = A_0$. This completes task $t(0) = (0, 0)$, by ensuring that $A_0$ embeds in $B_0$. See the general case below for how to define the tasks $(0, j)$ for $j > 0$.

At stage $n$, we are given $B_{n-1}$ and a task $t(n)$. If this task is a structure $A_i$, use JEP to embed $B_{n-1}$ and $A_i$ into a structure $B_n$ in $\mathcal{K}$. On the other hand, if the task is an embedding $g\colon C \to A_i$, where $C$ is a substructure of some $B_k$ with $k < n$, let $f\colon C \to B_{n-1}$ be the embedding obtained by restricting the embedding $B_k \to B_{n-1}$ to $C$. Then use AP to embed $B_{n-1}$ and $C$ into a structure $B_n$ in $\mathcal{K}$ in a way that makes the square over $C$ commute.

$$\begin{tikzcd}
B_k \arrow[r] & \dots \arrow[r] & B_n \\
C \arrow[u] \arrow[rr] & & A_i \arrow[u]
\end{tikzcd}$$

We finish stage $n$ by defining the tasks $(n, j)$ for $j > 0$. Enumerate the embeddings $g_j\colon C \to A_i$, where $C$ is a substructure of $B_n$ and $A_i$ is on our list of isomorphism representatives. There are countably many such embeddings, since $B_n$ has finitely many finite substructures $C$, there are countably many of the $A_i$, and for fixed $C$ and $A_i$, there are finitely many embeddings $C \to A_i$. Define task $(n, j)$ to be the embedding $g_{j-1}$.

After defining the entire sequence $(B_n)_{n\in\omega}$, we have also completed every task. Let $M = \varinjlim B_n$. Then $M$ is countable, since it is a directed colimit of finite structuers, and $\mathrm{age}(M) = \mathcal{K}$ exactly as in Proposition 6.1. It remains to check $\mathcal{K}$-homogeneity.

So suppose $f\colon C \to M$ and $g\colon C \to A$ are embeddings, with $A, C \in \mathcal{K}$. Then the image of $C$ in $M$ is contained in the image of some $B_n$, so $f = \sigma_n \circ f'$, for some $f'\colon C \to B_n$, where $\sigma_n\colon B_n \to M$ is the canonical embedding. After composing with isomorphisms, we can safely assume that $C$ is a substructure of $B_n$ and $A$ is $A_i$ for some $i$. So $g\colon C \to A$ was added as a task $(n, j) = t(m)$ for some $j$, and there is an embedding $A \to B_m$ making the square commute. Composing with the canonical embedding $\sigma_m\colon B_m \to M$ finishes the proof. $\square$

**Example 6.7.** Here are some examples of Fraïssé classes and their Fraïssé limit.

- The class of finite graphs. The Fraïssé limit is called the **random graph** (or the Rado graph), which we will discuss further below.

[17] The usual "diagonals" bijection, which enumerates $\omega \times \omega$ as $(0, 0)$, $(0, 1)$, $(1, 0)$, $(0, 2)$, $(1, 1)$, $(2, 0)$,... works.

- The class of finite triangle-free graphs. The Fraïssé limit is called the Henson graph.
- The class of finite linear orders. The Fraïssé limit is the countable dense linear order without endpoints $(\mathbb{Q}, <)$.
- The class of finite equivalence relations. The Fraïssé limit is the equivalence relation with countably infinitely many countably infinite classes.

**Example 6.8.** The simplest isomorphism-closed class of finite structures with HP and JEP but not AP is the class of acyclic graphs, also known as **forests**[18]. Let $A$ consist of two vertices $x$ and $y$, not connected by an edge. Let $B$ be $A$ together with a new vertex connecting $x$ and $y$ by a path of length 2. And let $C$ be $A$ together with two new vertices conncting $x$ and $y$ by a path of length 3. Then $B$ and $C$ cannot be amalgamated over $A$.

When the vocabulary is finite, for any finite structure $A$, enumerated $(a_i)_{i=1}^n$, there is a quantifier-free formula $\theta_A(x_1, \ldots, x_n)$ such that $B \models \theta_A(b_1, \ldots, b_n)$ if and only if the map $a_i \mapsto b_i$ is an embedding $A \to B$. This formula is essentially just the conjunction of all atomic formulas true in $A$ (see Proposition 4.10).

Now if $\mathcal{K}$ is a Fraïssé class, we can axiomatize the theory of Fraïssé limits of $\mathcal{K}$ as follows:

- $\forall$ axioms for $\mathcal{K}$: For every finite structure $A$ *not* in $\mathcal{K}$, let $\chi_A$ be
$$\forall x_1 \ldots \forall x_n \neg\theta_A(x_1, \ldots, x_n).$$
- $\forall\exists$ extension axioms: For every finite structure $B$ in $\mathcal{K}$ enumerated $(b_i)_{i=1}^n$ and every substructure $A \subseteq B$ which is an initial segment in the enumeration, $(b_i)_{i=1}^m$ for $0 \leq m \leq n$, let $\varphi_{A,B}$ be
$$\forall x_1 \ldots \forall x_m (\theta_A(x_1, \ldots, x_m) \to \exists x_{m+1} \ldots \exists x_n \, \theta_B(x_1, \ldots, x_n)).$$

We call $T_{\mathcal{K}}$, the set of all axioms of the above forms, the **generic theory of** $\mathcal{K}$. If $M \models T_{\mathcal{K}}$ and $M$ is countable, then $M$ is a Fraïssé limit of $K$.

- Since $M \models \chi_A$ for all $A \notin \mathcal{K}$, no structure not in $\mathcal{K}$ embeds in $M$, so $\text{age}(M) \subseteq \mathcal{K}$. The converse follows from $\mathcal{K}$-homogeneity, since for any structure $B$ in $\mathcal{K}$, the empty substructure of $B$ is isomorphic to the empty substructure of $M$ ($\mathcal{K}$ contains a unique empty structure, by HP and JEP), so $B$ embeds in $M$ over the empty structure.
- For $\mathcal{K}$-homogeneity, suppose $f \colon A \to M$ and $g \colon A \to B$ are embeddings with $A$ and $B$ in $\mathcal{K}$. We may assume $A$ is a substructure of $B$. Enumerate $B$ as $(b_i)_{i=1}^n$, ensuring that $A$ is the initial segment $(b_i)_{i=1}^m$. Then $M \models \theta_A(f(b_1), \ldots, f(b_m))$, and $M \models \varphi_{A,B}$, so there exists $c_{m+1}, \ldots, c_n$ such that $M \models \theta_B(f(b_1), \ldots, f(b_m), c_{m+1}, \ldots, c_n)$. Extend $f$ to a map $B \to M$ by $b_i \mapsto c_i$ for all $m < i \leq n$.

[18] A **tree** is a *connected* acyclic graph, and a forest is a disjoint union of trees

When $\mathcal{K}$ contains arbitrarily large finite structures, it follows that $T_{\mathcal{K}}$ is $\aleph_0$-categorical, and hence complete by Vaught's test.

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It is often possible to give a simpler axiomatization of $T_{\mathcal{K}}$ than the one described above. One simplification which is always possible is to assume in the extension axiom $\varphi_{A,B}$ that $|B| = |A| + 1$. We call such an axiom a **one-point extension axiom**. Indeed, if $M$ satisfies all one-point extension axioms, then it is easy to show that it is $\mathcal{K}$-homogeneous by induction on $|B| - |A|$, extending an embedding of $A$ into $M$ to $B$ one point at a time.

Here are some further examples of simplifications of $T_{\mathcal{K}}$.

**Example 6.9.** Let $\mathcal{K}_{\text{lo}}$ be the class of finite linear orders. The universal part of $T_{\mathcal{K}_{\text{lo}}}$ can be axiomatized by the theory of linear orders:

$$\forall x \forall y \forall z\, ((x < y \wedge y < z) \to x < z)$$
$$\forall x \forall y\, (x < y \vee y < x \vee x = y)$$
$$\forall x\, \neg(x < x)$$

Indeed, if $M$ satisfies the three sentences above, then already $\text{age}(M) \subseteq \mathcal{K}_{\text{lo}}$.

The $\forall\exists$ part of $T_{\mathcal{K}_{\text{lo}}}$ can be reduced to the following axioms:

$$\forall x \exists y\, x < y$$
$$\forall x \exists y\, y < x$$
$$\forall x \forall y (x < y \to \exists z\, (x < z \wedge z < y))$$

Indeed, if $M$ satisfies all six of the sentences above, then it satisfies all of the one-point extension axioms. For any finite subset $A \subseteq M$ and any embedding $A \to B$ where $|B| = |A| + 1$, the new element in $b$ is either greater than every point in $A$, less than every point in $A$, or between two points of $A$, and one of the three $\forall\exists$-sentences above handles each of those cases.

Of course, this verifies that the Fraïssé limit of $\mathcal{K}_{\text{lo}}$ is the countable dense linear order without endpoints.

**Example 6.10.** Let $\mathcal{G}$ be the class of finite graphs. The universal part of $T_{\mathcal{G}}$ can be axiomatized by the theory of graphs:

$$\forall x\, \neg(xEx)$$
$$\forall x \forall y\, (xEy \to yEx)$$

Indeed, if $M$ satisfies the two sentences above, then already $\text{age}(M) \subseteq \mathcal{G}$.

The $\forall\exists$ part of $T_{\mathcal{G}}$ can be reduced to the following schema, one sentence $\varphi_{m,n}$ for every pair $m, n \in \omega$:

$$\forall x_1 \dots \forall x_m \forall y_1 \dots \forall y_n \left( \left( \bigwedge_{i,j} x_i \neq y_j \right) \to \exists z \bigwedge_{1 \le i \le m} zEx_i \wedge \bigwedge_{1 \le j \le n} \neg zEy_j \right)$$

This is simpler than the general one-point extension axiom $\varphi_{A,B}$ since we don't have to describe the graph structure on $A = \{x_1, \dots, x_m, y_1, \dots, y_n\}$, we just need to know which elements of $A$ the new element $z$ connects to.

In the case of linear orders, we have a concrete representative of the Fraïssé limit, namely $(\mathbb{Q}, <)$. The Fraïssé limit of the class of finite graphs a bit more complicated. However, there are several pleasant presentations of this structure. Here are two of them:

- Consider the relation $E$ on $\omega$, defined by $mEn$ if and only if $n < m$ and the binary expansion of $m$ has a 1 in the $n^{\text{th}}$ position (the $2^n$ place), or $m < n$ and the binary expansion of $n$ has a 1 in the $m^{\text{th}}$ position (the $2^m$ place). Then $(\omega, E)$ is a Fraïssé limit of $\mathcal{G}$.

  Indeed, $E$ is a symmetric and irreflexive binary relation on $\omega$, and given any distinct numbers $x_1, \ldots, x_m, y_1, \ldots, y_n \in \omega$, we can find a natural number which is greater than all of these numbers and whose binary expansion ensures it is related to $x_1, \ldots, x_m$ and not related to $y_1, \ldots, y_n$.

- Build a graph with domain $\omega$ at random as follows: For every pair $\{m, n\}$ with $m \neq n$, flip a coin and put an edge between $n$ and $m$ if it comes up heads. That is, $mEn$ and $nEm$ holds with probability $1/2$, independently for all pairs $\{m, n\}$. This is called the Erdős–Rényi coin-flipping process. Then, with probability 1, $(\omega, E)$ is a Fraïssé limit of $\mathcal{G}$. This is why the Fraïssé limit of $\mathcal{G}$ is called the **random graph**.

  We can check this with a simple probabilistic argument. Consider a fixed one-point extension axiom $\varphi_{m,n}$. Suppose we have distinct elements $x_1, \ldots, x_m, y_1, \ldots, y_n$ from $\omega$. For any $z$ not among these elements, the probability that $z$ is a witness for the extension axiom (i.e. $z$ is connected to the $x_i$ and not connected to the $y_j$) is exactly $2^{-(m+n)}$. And crucially, the probabilities that $z$ and $z'$ are witnesses are independent. Since there are infinitely many $z$ not among the $x_i$ and $y_j$, we have infinitely many independent chances to find a witness, each with positive probability. By the Borel-Cantelli Lemma, we find infinitely many witnesses in $\omega$ with probability 1. Now there are countably many sequences of $x_i$ and $y_j$, there are countably many extension axioms, and the intersection of countably many probability 1 events has probability 1. So $(\omega, E) \models T_{\mathcal{G}}$ with probability 1.

A variant of the probabilistic argument above gives a remarkable fact: the first-order zero-one law for the class of finite graphs.

Let $\mathcal{K}$ be an isomorphism-closed class of finite structures which contains at least one structure of size $n$ for each $n \in \omega$. Write $\mathcal{K}_{[n]}$ for the set of structures in $\mathcal{K}$ with domain $[n] = \{1, \ldots, n\}$. We write

$$\mathbb{P}_n(\varphi) = \frac{|\{A \in \mathcal{K}_{[n]} \mid A \models \varphi\}|}{|\mathcal{K}_{[n]}|}$$

for the probability that a structure $A$ in $\mathcal{K}_{[n]}$, selected uniformly at random, satisfies $\varphi$. The **asymptotic probability** of $\varphi$ is $\lim_{n\to\infty} \mathbb{P}_n(\varphi)$, if this limit exists.

The **almost-sure theory of** $\mathcal{K}$ is the set of all sentences with asymptotic probability 1. This is the set of first-order properties of structures in $\mathcal{K}$ that hold of almost all large finite structures in $\mathcal{K}$.

**Lemma 6.11.** *The almost-sure theory $T_{as}$ of $\mathcal{K}$ is consistent and closed under entailment.*

*Proof.* We first show that $T_{\mathrm{as}}$ is closed under entailment. Suppose $T_{\mathrm{as}} \models \theta$. Then there are finitely many sentences $\psi_1, \ldots, \psi_k \in T_{\mathrm{as}}$ such that $\bigwedge_{i=1}^k \psi_i \models \theta$. Since each $\psi_i$ has asymptotic probability 1, for any $\varepsilon > 0$, there is some $N$ such that for all $n > N$ and all $1 \leq i \leq k$,

$$\mathbb{P}_n(\psi_i) \geq 1 - \varepsilon/k.$$

Since $\mathbb{P}_n(\neg\psi_i) \leq \varepsilon/k$, we have $\mathbb{P}_n(\neg \bigwedge_{i=1}^k \psi_i) \leq \varepsilon$. And since $\bigwedge_{i=1}^k \psi_i \models \theta$, it follows that

$$\mathbb{P}_n(\theta) \geq \mathbb{P}_n\left(\bigwedge_{i=1}^k \psi_i\right) \geq 1 - \varepsilon.$$

So $\lim_{n\to\infty} \mathbb{P}_n(\theta) = 1$, and $\theta \in T_{\mathrm{as}}$.

It follows that $T_{\mathrm{as}}$ is consistent. If not, then $T_{\mathrm{as}} \models \bot$, so $\bot \in T_{\mathrm{as}}$, and $\lim_{n\to\infty} \mathbb{P}_n(\bot) = 1$, but $\mathbb{P}_n(\bot) = 0$ for all $n$, which is a contradiction. □

We say that $\mathcal{K}$ has a **first-order zero-one law** if for every sentence $\varphi$,

$$\lim_{n\to\infty} \mathbb{P}_n(\varphi) = 0 \text{ or } 1.$$

Equivalently, $\mathcal{K}$ has a first-order zero-one law if and only if its almost-sure theory is complete.

Thursday 11/1 **Theorem 6.12.** *Let $\mathcal{G}$ be the class of all finite graphs. Then $\mathcal{G}$ has a first-order zero-one law, and its almost-sure theory is equivalent to $T_{\mathcal{G}}$, the theory of the random graph.*

*Proof.* The proof has two components: a logical argument and a probabilistic argument. We deal with the logical argument first.

*Claim:* It suffices to show that every one-point extension axiom $\varphi_{m,n}$ has asymptotic probability 1.

Let $T_{\mathrm{as}}$ be the almost-sure theory of $\mathcal{G}$. Since the axioms of the theory of graphs hold of all structures in $\mathcal{G}$, these sentences have asymptotic probability 1. Assuming also that every one-point extension axiom has asymptotic probability 1, we have $T_{\mathcal{G}} \subseteq T_{\mathrm{as}}$. By Lemma 6.11, every consequence of $T_{\mathcal{G}}$ is in $T_{\mathrm{as}}$.

But since $T_{\mathcal{G}}$ is complete and $T_{\mathrm{as}}$ is consistent, $T_{\mathrm{as}}$ must be equal to the set of consequences of $T_{\mathcal{G}}$. So $T_{\mathrm{as}}$ is complete, and $\mathcal{G}$ has a first-order zero-one law.

Having established the claim, we fix a one-point extension axiom $\varphi_{m,n}$, and let $m + n = k$. Observe that the uniform probability measure $\mathbb{P}_N$ on $\mathcal{G}_{[N]}$ agrees with the Erdős–Rényi coin-flipping process: for any $\{i, j\} \subseteq [N]$ with $i \neq j$, exactly half of the graphs in $\mathcal{G}_{[N]}$ have $iEj$, and these probabilities are independent between pairs.

Now fix some distinct $x_1, \ldots, x_m, y_1, \ldots, y_n \in [N]$. The probability that some $z$ is a witness to the extension axiom $\varphi_{m,n}$ for these $x_i$ and $y_j$ is $2^{-(m+n)}$.

Define $\delta = 2^{-(m+n)} > 0$. There are $N - k$ possible witnesses $z$, and each is a witness with probability $\delta$, independently. So the probability that there is *no* witness to the extension axiom over the $x_i$ and $y_j$ is $(1-\delta)^{N-k}$. It follows that

$$\mathbb{P}_N(\neg\varphi_{m,n}) \leq N^k(1-\delta)^{N-k},$$

since $N^k$ is an upper bound for the number of choices of the $x_i$ and $y_j$.

Now when we compute $\lim_{N\to\infty} \mathbb{P}_N(\neg\varphi_{m,n})$, $k$ is fixed and $N$ grows, so the polynomial $N^k$ is dominated by the exponential decay $(1-\delta)^{N-k}$ (since $1-\delta < 1$), and $\lim_{N\to\infty} \mathbb{P}_N(\neg\varphi_{m,n}) = 0$. Hence, $\varphi_{m,n}$ has asymptotic probability 1. □

**Example 6.13.** To see the necessity of considering properties expressible by first-order sentences, note that the following limit does not exist:

$$\lim_{n\to\infty} \frac{|\{A \in \mathcal{K}_{[n]} \mid |A| \text{ is even}\}|}{|\mathcal{K}_{[n]}|}.$$

Indeed, the property "$|A|$ is even" is not expressible by a first-order sentence relative to the theory of graphs.

We have also seen that the property "$A$ is connected" is not expressible by a first-order sentence relative to the theory of graphs. However, the random graph is connected with diameter 2, since it satisfies $\forall x \forall y \exists z\, (xEz \wedge zEy)$. It follows that this sentence is in the almost-sure theory of $\mathcal{G}$: almost all large finite graphs have diameter 2, so almost all large finite graphs are connected.

**Exercise 24.** Let $\mathcal{K}$ be the class of all finite $\mathcal{V}$-structures, for any single-sorted finite relational vocabulary $\mathcal{V}$.

(a) Show that $\mathcal{K}$ is a Fraïssé class, and axiomatize its generic theory $T_{\mathcal{K}}$.

(b) Show that $\mathcal{K}$ has a first-order zero-one law, and its almost-sure theory is $T_{\mathcal{K}}$ (the argument will be very similar to the argument for the random graph).

When there are function symbols in the language, we typically do not have a first-order zero-one law, as the following exercise demonstrates.

**Exercise 25.** Let $\mathcal{V}$ be the vocabulary with a single unary function symbol $f$. Let $\mathcal{K}$ be the class of all finite $\mathcal{V}$ structures. Let $\varphi$ be the sentence $\forall x\, (f(x) \neq x)$. Show that

$$\lim_{n\to\infty} \mathbb{P}_n(\varphi) = \frac{1}{e}.$$

**Example 6.14.** Let $\mathcal{G}_\triangle$ be the class of all finite triangle-free graphs. Erdős, Kleitman, and Rothschild showed that almost all triangle-free graphs are bipartite, i.e.

$$\lim_{n\to\infty} \frac{|\{A \in (\mathcal{G}_\triangle)_{[n]} \mid A \text{ is bipartite}\}|}{|(\mathcal{G}_\triangle)_{[n]}|} = 1.$$

Kolaitis, Promel, and Rothschild refined this result, showing that $\mathcal{G}_\triangle$ has a first-order zero-one law, and its almost-sure theory is equal to the generic theory of bipartite graphs. To be more precise, the class of all finite bipartite graphs in the vocabulary of graphs is not a Fraïssé class (it fails amalgamation, since paths of lengths 2 and 3 cannot be amalgamated over their endpoints). But if we view bipartite graphs in the vocabulary $\mathcal{V}_{\mathrm{bi}} = (E, L, R)$, where $L$ and $R$ are unary relations naming the bipartition, then the class $\mathcal{G}_{\mathrm{bi}}$ of finite bipartite graphs is a Fraïssé class, with a generic theory $T_{\mathcal{G}_{\mathrm{bi}}}$. And the theorem is that the almost-sure theory of $\mathcal{G}_\triangle$ is equal to the subtheory of $T_{\mathcal{G}_{\mathrm{bi}}}$ consisting of those sentences which do not mention $L$ and $R$.

In particular, the almost-sure theory of $\mathcal{G}_\triangle$ is not equal to the generic theory $T_{\mathcal{G}_\triangle}$, since the Fraïssé limit of $\mathcal{G}_\triangle$ contains odd-length cycles (for example, the pentagon), which cannot be embedded in any bipartite graph. This raises an interesting question (which is a major open problem at the border of model theory and combinatorics): We say that a theory has the **finite model property** if every sentence in the theory has a finite model. The theory of the random graph, $T_{\mathcal{G}}$, has the finite model property (in fact, almost all finite graphs are models, by the zero-one law). Does $T_{\mathcal{G}_\triangle}$ have the finite model property?

It is even an open question whether there exists a finite triangle-free graph satisfying the one-point extension axioms over subsets of size at most 4.

## 6.2 Quantifier elimination

A theory $T$ has **quantifier elimination** if every formula is $T$-equivalent to a quantifier-free formula.

This notion is extremely important in "applied" model theory: If we want to understand a given first-order theory, the first step is to understand the definable sets, i.e. the formulas up to $T$-equivalence. And in the best-case scenario, when $T$ has quantifier elimination, we find that every definable set is a Boolean combination of sets defined by atomic formulas, which are hopefully easy to understand.

For $M \models T$ and $a \in M^x$, define

$$\mathrm{qftp}(a) = \{\varphi(x) \mid \varphi \text{ is quantifier-free and } M \models \varphi(a)\}.$$

Note that $\mathrm{qftp}(a) \subseteq \mathrm{tp}(a)$. Intuitively, $\mathrm{qftp}(a)$ describes the isomorphism-type of the substructure $\langle a \rangle$ of $M$ generated by $a$.

**Proposition 6.15.** *Let $a \in M^x$ and $a' \in (M')^x$, where $M$ and $M'$ are models of $T$. Then $\mathrm{qftp}(a) = \mathrm{qftp}(a')$ if and only if $\langle a \rangle \cong \langle a' \rangle$ by an isomorphism mapping $a$ to $a'$.*

*Proof.* Let $A = \langle a \rangle$ and $A' = \langle a' \rangle$. Suppose $f\colon A \to A'$ is an isomorphism with $f(a) = a'$. Let $\varphi(x)$ be a quantifier-free formula. Then $\varphi(x)$ is both universal and existential, so it is preserved and reflected by embeddings (in particular, the inclusions $A \to M$ and $A' \to M'$ and the isomorphism $f\colon A \to A'$). Then:

$$\begin{aligned} M \models \varphi(a) &\iff A \models \varphi(a) \\ &\iff A' \models \varphi(f(a)) \\ &\iff M' \models \varphi(a'). \end{aligned}$$

So $\mathrm{qftp}(a) = \mathrm{qftp}(a')$.

Conversely, suppose $\mathrm{qftp}(a) = \mathrm{qftp}(a')$. Recall (Exercise 1), that $b \in A = \langle a \rangle$ if and only if there is some term $t(x)$ such that $b = t^M(a)$, and similarly for $A' = \langle a' \rangle$ (of course, the term $t$ need not be unique).

Define a map $f \colon A \to A'$ by $f(t^M(a)) = t^{M'}(a')$. This is well-defined, since if $t^M(a) = (t')^M(a)$, then the formula $t(x) = t'(x)$ is in $\mathrm{qftp}(a) = \mathrm{qftp}(a')$, so $t^{M'}(a') = (t')^{M'}(a')$.

1. $f$ is injective: If $t^M(a) \neq (t')^M(a)$, then $t(x) \neq t'(x)$ is in $\mathrm{qftp}(a')$, so $t^{M'}(a') \neq (t')^{M'}(a')$.
2. $f$ is surjective: For any $b \in A'$, $b = t^{M'}(a')$ for some term $t(x)$, and $b = f(t^M(a))$.
3. $f$ maps $a$ to $a'$. Any element $a_i$ from the tuple $a$ is equal to $t^M(a)$ where $t(x)$ is the variable $x_i$. Then $f(a_i) = t^{M'}(a') = a'_i$.
4. $f$ respects functions symbols. For any function symbol $g$ and terms $t_1(x), \ldots, t_n(x)$ of the appropriate types, let $t(x)$ be the composite term $g(t_1, \ldots, t_n)$. Then we have
$$\begin{aligned} f(g^M(t_1^M(a), \ldots, t_n^M(a))) &= f(t^M(a)) \\ &= t^{M'}(a') \\ &= g^{M'}(t_1^{M'}(a'), \ldots, t_n^{M'}(a')) \\ &= g^{M'}(f(t_1^M(a)), \ldots, f(t_n^M(a))). \end{aligned}$$
5. $f$ preserves and reflects relation symbols. For any relation symbol $R$ and terms $t_1(x), \ldots, t_n(x)$ of the appropriate types, let $\varphi(x)$ be the quantifier-free formula $R(t_1, \ldots, t_n)$. Then we have
$$\begin{aligned} (t_1^M(a), \ldots, t_n^M(a)) \in R^M &\iff \varphi(x) \in \mathrm{qftp}(a) \\ &\iff \varphi(x) \in \mathrm{qftp}(a') \\ &\iff (t_1^{M'}(a'), \ldots, t_n^{M'}(a')) \in R^{M'} \\ &\iff (f(t_1^M(a)), \ldots, f(t_n^M(a))) \in R^M. \quad \square \end{aligned}$$

Our first test for quantifier elimination is simple: if complete types are determined by quantifier-free types, then $T$ has quantifier-elimination.

**Proposition 6.16.** *Suppose that for all contexts $x$ and all complete types $p$ and $q$ in $S_x(T)$, if $p$ and $q$ contain the same quantifier-free formulas, then $p = q$. Then $T$ has quantifier elimination.*

*Proof.* Let $\varphi(x)$ be any formula, and consider complete types $p$ and $q$ such that $\varphi(x) \in p$ and $\varphi(x) \notin q$. Then $p \neq q$, so $p$ and $q$ do not contain the same quantifier-free formulas. Since the quantifier-free formulas are closed under negation, we may assume there is some quantifier-free formula $\psi(x)$ such that $\psi(x) \in p$ and $\psi(x) \notin q$. By Corollary 5.8, $\varphi(x)$ is $T$-equivalent to a quantifier-free formula. $\square$

As an application of this test, we can show easily that theories of Fraïssé limits eliminate quantifiers.

**Corollary 6.17.** *Let $\mathcal{V}$ be a single-sorted finite relational vocabulary, and let $\mathcal{K}$ be a Fraïssé class of $\mathcal{V}$-structures. The generic theory $T_{\mathcal{K}}$ has quantifier elimination.*

*Proof.* We apply Proposition 6.16. Suppose $p$ and $q$ are complete types in $S_x(T_{\mathcal{K}})$ which contain the same quantifier-free formulas. Let $M \models T_{\mathcal{K}}$ be a countable model and $a, a' \in M^x$ such that $M \models p(a)$ and $M \models q(a')$. (How? Find a realization $a$ of $p$ in $M_1 \models T_{\mathcal{K}}$, and then find a realization $a'$ of $q$ in an elementary extension $M_1 \preceq M_2$ by Proposition 5.3, and finally take a countable elementary substructure $M \preceq M_2$ containing $a$ and $a'$ by Löwenheim–Skolem.)

Now $M$ is a Fraïssé limit of $K$. And since $p$ and $q$ contain the same quantifier-free formulas, $\langle a \rangle$ and $\langle a' \rangle$ are isomorphic by an isomorphism mapping $a$ to $a'$. By strong $\mathcal{K}$-homogeneity, this isomorphism extends to an automorphism $\sigma \colon M \to M$ such that $\sigma(a) = a'$. But then $\mathrm{tp}(a) = \mathrm{tp}(a')$, so $p = q$. By Proposition 6.16, $T_{\mathcal{K}}$ has quantifier elimination. $\square$

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We will now develop a slightly more refined test for quantifier-elimination. First, we reduce the problem on the syntactic side, then we prove a semantic equivalence.

A formula $\varphi(x)$ is **primitive existential** if it has the form $\exists y\, \psi(x, y)$, where $y$ is a single variable and $\psi(x, y)$ is a conjunction of atomic and negated atomic formulas.

**Lemma 6.18.** *Suppose that every primitive existential formula is $T$-equivalent to a quantifier-free formula. Then $T$ has quantifier elimination.*

*Proof.* We prove by induction that every formula is $T$-equivalent to a quantifier-free formula. This is clear for atomic formulas, and the induction steps for Boolean combinations are trivial. So it suffices to consider the case of a formula $\exists y\, \psi(x, y)$. By the inductive hypothesis, $\psi(x, y)$ is $T$-equivalent to a quantifier-free formula $\varphi(x, y)$, which is logically equivalent to a formula in disjunctive normal form (Exercise 19),

$$\bigvee_{i=1}^{n} \varphi_i(x, y),$$

where each $\varphi_i$ is a conjunction of atomic and negated atomic formulas. So $\exists y\, \psi(x, y)$ is $T$-equivalent to

$$\exists y \bigvee_{i=1}^{n} \varphi_i(x, y),$$

which is logically equivalent to

$$\bigvee_{i=1}^{n} \exists y\, \varphi_i(x, y).$$

Now each formula $\exists y\, \varphi_i(x, y)$ is primitive existential, so by assumption it is $T$-equivalent to a quantifier-free formula $\chi_i(x, y)$, and our original formula is $T$-equivalent to $\bigvee_{i=1}^{n} \chi_i(x, y)$. $\square$

Let $M$ and $M'$ be $\mathcal{V}$-structures, and suppose $A \subseteq M$. A map $f\colon A \to M'$ is **partial elementary** if for all formulas $\varphi(x)$ and all $a \in A^x$, $M \models \varphi(a)$ if and only if $M' \models \varphi(f(a))$.

**Theorem 6.19.** *The following are equivalent:*

(1) *$T$ has quantifier elimination.*

(2) *For any models $M \models T$ and $M' \models T$, if $A$ is a substructure of $M$ and $f\colon A \to M'$ is an embedding, then $f$ is partial elementary.*

(3) *For any models $M \models T$ and $M' \models T$, if $A$ is a finitely generated substructure of $M$, $f\colon A \to M'$ is an embedding, and $\varphi(x)$ is a primitive existential formula, then for all $a \in A^x$, if $M \models \varphi(a)$, then $M' \models \varphi(f(a))$.*

*Proof.* (1)⇒(2): Suppose $T$ has quantifier elimination and $f\colon A \to M'$ is an embedding, where $A$ is a substructure of $M \models T$, and $M' \models T$. Let $\varphi(x)$ be a formula and $a \in A^x$. Then $\varphi(x)$ is $T$-equivalent to a quantifier-free formula $\psi(x)$. Since $\psi(x)$ is both universal and existential, it is preserved and reflected by embeddings. So

$$\begin{aligned} M \models \varphi(a) &\iff M \models \psi(a) \\ &\iff A \models \psi(a) \\ &\iff M' \models \psi(f(a)) \\ &\iff M' \models \varphi(f(a)). \end{aligned}$$

(2)⇒(3): Trivial from the definition of partial elementary map.

(3)⇒(1): By Lemma 6.18, it suffices to show that every primitive existential formula is equivalent to a quantifier-free formula. We apply the separation lemma for formulas, Corollary 5.8. So let $\varphi(x)$ be a primitive existential formula, and let $p(x)$ and $q(x)$ be complete types such that $\varphi(x) \in p(x)$ and $\varphi(x) \notin q(x)$. Suppose for contradiction that no quantifier-free formula separates $p$ from $q$, i.e. (since quantifier-free formulas are closed under negation) that $p$ and $q$ contain the same quantifier-free formulas.

Let $a \in M^x$ and $a' \in (M')^x$ with $M, M' \models T$, $M \models p(a)$, and $M' \models q(a')$. Let $A = \langle a \rangle$ and $A' = \langle a' \rangle$. In particular, $A$ is a finitely generated substructure of $M$. Since $p$ and $q$ contain the same quantifier-free formulas, $\mathrm{qftp}(a) = \mathrm{qftp}(a')$, so $A \cong A'$ by an isomorphism mapping $a$ to $a'$. We may view this isomorphism as an embedding $f\colon A \to M'$. But then $\varphi(x) \in p(x)$ implies $M \models \varphi(a)$, so by assumption $M' \models \varphi(a')$, and $\varphi(x) \in q(x)$, contradiction. $\square$

As an application, we prove that the theory of algebraically closed fields has quantifier elimination.

**Theorem 6.20.** *ACF has quantifier elimination.*

*Proof.* We use the test in Theorem 6.19. So suppose $K$ and $K'$ are algebraically closed fields, $A \subseteq K$ is a finitely generated substructure, and $f\colon A \to K'$ is an embedding. Let $\exists y\, \varphi(x, y)$ be a primitive existential formula, and let $a \in A^x$ such that $K \models \exists y\, \varphi(a, y)$.

Let $B = f(A)$, so $f$ can be viewed as an isomorphism $A \cong B$. Now $A$ and $B$ are subrings of $K$ and $K'$, respectively. Let $A_1$ and $B_1$ be the subfields of $K$ and $K'$ generated by $A$ and $B$, respectively. So $A_1 \cong \mathrm{Frac}(A)$ and $B_1 \cong \mathrm{Frac}(B)$, and $f$ extends to an isomorphism $f_1\colon A_1 \cong B_1$.

Let $A_2$ and $B_2$ be the relative algebraic closures of $A_1$ and $B_1$ inside $K$ and $K'$, respectively. Since $K$ and $K'$ are algebraically closed, $A_2$ and $B_2$ are isomorphic to the algebraic closures of $A_1$ and $B_1$, and $f_1$ extends to an isomorphism $F_2\colon A_2 \cong B_2$.

Now let $b \in K$ such that $K \models \varphi(a, b)$. Let $A_3 = A_2(b)$, the subfield of $K$ generated by $A_2$ and $b$, and let $K''$ be a proper elementary extension of $K'$ (this exists because $K'$ is infinite, i.e. not boring). We claim that we can find an embedding $g\colon A_2(b) \to K''$.

*Case 1:* $b \in A_2$. Then $A_2(b) = A_2$, and we can take $g = f_2$ (composed with the inclusions $B_2 \subseteq K' \preceq K''$).

*Case 2:* $b \notin A_2$. Then since $A_2$ is relatively algebraically closed in $K$, the field $A_2(b)$ is a transcendental extension of $A_2$, isomorphic to $A_2(x)$. Since $K''$ is a proper extension of $B_2$, and $B_2$ is algebraically closed, any $b' \in K'' \setminus B_2$ is transcendental over $B_2$. So $B_2(b')$ is isomorphic to $B_2(x)$, and $f_2$ extends to an isomorphism $f_3\colon A_2(b) \cong B_2(b')$, which we view as an embedding $A_2(b) \to K''$.

Note that we only needed the elementary extension $K''$ to handle the case when $K'$ doesn't already contain an element transcendental over $B_2$, i.e. when $K' = B_2$.

$$\begin{tikzcd}
K & K'' \\
A_2(b) \arrow[u, dash] \arrow[ur, dashed, "f_3"] & K' \arrow[u, dashed, dash, "\preceq"'] \\
A_2 \arrow[u, dash] \arrow[r, "f_2"] & B_2 \arrow[u, dash] \\
A_1 \arrow[u, dash] \arrow[r, "f_1"] & B_1 \arrow[u, dash] \\
A \arrow[u, dash] \arrow[r, "f"] & B \arrow[u, dash]
\end{tikzcd}$$

To finish, observe that since $K \models \varphi(a, b)$, and $\varphi(x, y)$ is quantifier-free, we have $A_2(b) \models \varphi(a, b)$, so $K'' \models \varphi(f_3(a), f_3(b))$, so $K'' \models \exists y\, \varphi(f(a), y)$, and since $K' \preceq K''$, also $K' \models \exists y\, \varphi(f(a), y)$, as desired. $\square$

Let's think about what this theorem means in context. The vocabulary of rings has no relation symbols, so every atomic formula has the form $t(x) = t'(x)$, where $t$ and $t'$ are terms in context $x$. Modulo the theory of rings, this formula is equivalent to $t(x) - t'(x) = 0$, and each term can be rewritten as a polynomial, so every atomic formula is equivalent to one of the form $p(x) = 0$, where $p$ is a polynomial. The definable set $\{x \in K^n \mid p(x) = 0\}$ is a closed set in the Zariski topology on $K^n$. On the other hand, the negated atomic formula $p(x) \neq 0$ defines a Zariski open set.

Now a quantifier-free formula $\varphi(x)$ defines a Boolean combination of Zariski closed sets. Such a set is called a **constructible set**[19].

As we have noted before, the operation of adding an existential quantifier to a formula corresponds to the coordinate projection of definable sets. For example, the projection of the set $\{(x, y) \in K^2 \mid xy = 1\}$ onto the first coordinate is the set $\{x \in K \mid x \neq 0\}$. This corresponds to the fact that the formula $\exists y\, (xy = 1)$ is ACF-equivalent to the quantifier-free formula $x \neq 0$.

Keeping this in mind, we find that quantifier elimination for ACF is equivalent to (a form of) Chevalley's theorem on constructible sets. Of course, there are fancier scheme-theoretic versions of this theorem in algebraic geometry.

**Corollary 6.21** (Chevalley's theorem)**.** *A coordinate projection of a Zariski constructible set is Zariski constructible.*

Quantifier elimination can also be a tool for proving completeness.

**Proposition 6.22.** (1) *If $T$ is complete, then all models of $T$ have isomorphic minimal substructures. That is, for any $M, M' \models T$, $\langle\emptyset\rangle_M \cong \langle\emptyset\rangle_{M'}$.*

(2) *The converse is true if $T$ has quantifier elimination.*

(3) *If $\mathcal{V}$ has no 0-ary function symbols (constant symbols) or 0-ary relation symbols (proposition symbols), then quantifier elimination implies completeness.*

*Proof.* For (1), suppose $T$ is complete. Then the quantifier-free type of the empty tuple in $M \models T$ is the set of all quantifier-free sentences true in $M$. Since $T$ is complete, every quantifier-free sentence true in $M$ is entailed by $T$. So if $M, M' \models T$, $\mathrm{qftp}_M(\emptyset) = \mathrm{qftp}_{M'}(\emptyset)$. Then $\langle\emptyset\rangle_M \cong \langle\emptyset\rangle_{M'}$ by Proposition 6.15.

For (2), suppose that $T$ has quantifier elimination and all models of $T$ have isomorphic minimal substructures. Suppose for contradiction that $T$ is not complete. Then there is a sentence $\varphi$ and models $M \models T \cup \{\varphi\}$ and $M' \models T \cup \{\neg\varphi\}$. But the isomorphism $\langle\emptyset\rangle_M \to \langle\emptyset\rangle_{M'}$ is a partial elementary map by Theorem 6.19, so $M \models \varphi$ if and only if $M' \models \varphi$, contradiction.

For (3), note that if $\mathcal{V}$ has no constant symbols, then $\langle\emptyset\rangle_M$ is empty for all $\mathcal{V}$-structures $M$. And if $\mathcal{V}$ has no proposition symbols, then there is a unique empty $\mathcal{V}$-structure up to isomorphism. It follows that for any $M, M' \models T$, $\langle\emptyset\rangle_M \cong \langle\emptyset\rangle_{M'}$.

[19] Sometimes a constructible set is defined to be a finite union of locally closed sets, i.e. sets which are open in their closure. This is equivalent: a locally closed set is the intersection of an open set and a closed set, and any Boolean combination of closed sets can be written as a finite union of locally closed sets, thanks to the disjunctive normal form

Another way of proving (3) is to note that if $T$ has quantifier-elimination, then every sentence is $T$-equivalent to a quantifier-free sentence. But if there are no constant symbols or proposition symbols, the only quantifier-free sentences are $\top$ and $\bot$. $\square$

This gives us a new proof that for fixed $p$, either prime or 0, the theory $\mathrm{ACF}_p$ is complete. Indeed, since $\mathrm{ACF} \subseteq \mathrm{ACF}_p$, $\mathrm{ACF}_p$ has quantifier elimination. And for any $M, M' \models \mathrm{ACF}_p$, $\langle \emptyset \rangle_M \cong \langle \emptyset \rangle_{M'} \cong \mathbb{F}_p$.

If a theory $T$ has a reasonable (meaning computable) axiomatization, then as soon as we know $T$ is complete, we have an algorithm for deciding which sentences are entailed by $T$. For any sentence $\varphi$, start searching for a proof $T \vdash \varphi$, and simultaneously start searching for a proof $T \vdash \neg\varphi$. Since $T$ is complete, one of these searches eventually terminates.

But the algorithm described above is hopelessly inefficient. In many situations, a more efficient algorithm can be found via effective quantifier elimination, i.e. an algorithm for finding a quantifier-free formula which is $T$-equivalent to a given formula. Applying this algorithm to an arbitrary sentence $\varphi$ produces a quantifier-free sentence $\psi$ which is $T$-equivalent to it. But a quantifier-free sentence $\psi$ is just a Boolean combination of atomic statements about $\langle \emptyset \rangle$, the truth value of which can be easily checked.

The proof we gave above that ACF has quantifier elimination was entirely ineffective. But an effective quantifier elimination algorithm exists for ACF; in this case, a quantifier-free sentences comes down to an explicit list of the characteristics $p$ in which the sentence is true. And effective quantifier elimination algorithms also exist for other interesting theories, like $\mathrm{Th}(\mathbb{R}; 0, 1, +, -, \cdot, \leq)$, the theory of real closed ordered fields (this is know as Tarski's Theorem).

**Exercise 26.** Let $s \colon \omega \to \omega$ be the successor function $n \mapsto n + 1$. Show that $\mathrm{Th}(\omega; s)$ does not have quantifier elimination, but $\mathrm{Th}(\omega; s, 0)$ does (where 0 is a constant symbol naming the element 0).

**Exercise 27.** Let $K$ be a field, and let $T$ be the theory of infinite vector spaces over $K$ in the single-sorted vocabulary of Example 1.2. Show that $T$ has quantifier elimination.

**Exercise 28.** Let $T$ be the theory of infinite-dimensional vector spaces over algebraically closed fields of characteristic 0 in the two-sorted vocabulary of Example 1.1. That is, $T$ consists of the vector space axioms, the axioms of $\mathrm{ACF}_0$ in the field sort $k$, and axioms asserting that the vector space $v$ is not $n$-dimensional for any $n$ (if it is not clear to you how to axiomatize this theory, then you can consider it part of the exercise). Show that $T$ does *not* have quantifier elimination.

## 6.3 Model completeness and model companions

Thursday 11/8 One advantage of showing that $T$ has quantifier elimination is that the notions of elementary embedding and elementary substructure become much simpler. If $T$

has quantifier elimination, then every embedding between models of $T$ is an elementary embedding (every formula is $T$-equivalent to a quantifier-free formula, and quantifier-free formulas are preserved and reflected by embeddings).

Some theories have this nice property, without having quantifier elimination in full. A theory $T$ is **model complete** if every embedding between models of $T$ is an elementary embedding.

Part (a) of the following exercise explains the name "model complete". Part (b) shows that quantifier elimination could also be called "substructure completeness".

**Exercise 29.** (a) Show that $T$ is model complete if and only if for any model $M \models T$, $T \cup \mathrm{Diag}(M)$ is a complete $\mathcal{V}(M)$-theory.

(b) Show that $T$ has quantifier elimination if and only if for any model $M \models T$, and any substructure $A \subseteq M$, $T \cup \mathrm{Diag}(A)$ is a complete $\mathcal{V}(A)$-theory.

**Proposition 6.23.** *Every model complete theory $T$ has a $\forall\exists$ axiomatization, i.e. $T$ is equivalent to a $\forall\exists$-theory.*

*Proof.* By Theorem 5.15, it suffices to show that $T$ is preserved by directed colimits. So suppose $(M_i)_{i \in I}$ is a directed system of $\mathcal{V}$-structures, such that $M_i \models T$ for all $i \in I$, and let $M = \varinjlim M_i$. Since $T$ is model complete, this is a directed colimit along elementary embeddings, so by Proposition 5.13, each map $\sigma_i \colon M_i \to M$ is an elementary embedding. In particular, $M \models T$. $\square$

**Example 6.24.** In 1996, Wilkie showed that $\mathrm{Th}(\mathbb{R}; 0, 1, +, -, \cdot, e^x)$, where $e^x$ is viewed as a unary function symbol, is model complete. It follows that this theory has a $\forall\exists$ axiomatization, but we don't know what it is. In fact, giving an explicit axiomatization of this theory would almost certainly require proving the **real Schanuel's conjecture**: If $x_1, \ldots, x_n$ are real numbers such that $\mathbb{Q}(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n})$ has transcendence degree less than $n$ over $\mathbb{Q}$, then $x_1, \ldots, x_n$ are linearly dependent over $\mathbb{Q}$.

Suppose $M \subseteq N$ are $\mathcal{V}$-structures. We say that $M$ is **existentially closed in** $N$ if for any $\exists$-formula $\varphi(x)$ and any $a \in M^x$, if $N \models \varphi(a)$, then $M \models \varphi(a)$. We say that a model $M \models T$ is **existentially closed** (in models of $T$) if whenever $M \subseteq N$ and $N \models T$, $M$ is existentially closed in $N$.

**Theorem 6.25.** *The following are equivalent:*

(1) *$T$ is model complete.*

(2) *Every model of $T$ is existentially closed.*

(3) *Every formula is $T$-equivalent to a $\forall$-formula.*

(4) *Every formula is $T$-equivalent to an $\exists$-formula.*

*Proof.* (1)⇒(2): If $M, N \models T$ and $M \subseteq N$, then since $T$ is model complete, $M \preceq N$, so $M$ is existentially closed in $N$.

(2)⇒(3): First note that (2) implies that all $\exists$-formulas are reflected by embeddings between models of $T$. Indeed, if $f\colon M \to N$ is an embedding, with $M, N \models T$, then $M \cong f(M) \subseteq N$, and $f(M)$ is existentially closed in $N$. So if $N \models \varphi(f(a))$, where $\varphi(x)$ is an $\exists$-formula, then $f(M) \models \varphi(f(a))$, and $M \models \varphi(a)$.

So by Theorem 5.11, every $\exists$-formula is $T$-equivalent to a $\forall$-formula.

We now prove that every formula is $T$-equivalent to a $\forall$-formula. By definition, the class of $\forall$-formulas contains the atomic formulas and $\top$ and $\bot$ and is closed under $\wedge$, $\vee$, and $\forall$. So it suffices to show that up to $T$-equivalence, the class of $\forall$-formulas is closed under $\neg$ and $\exists$.

For $\neg$, if $\varphi(x)$ is a $\forall$-formula, then $\neg\varphi(x)$ is an $\exists$-formula, which is $T$-equivalent to a $\forall$-formula by the observation above. For $\exists$, if $\varphi(x, y)$ is a $\forall$-formula, then $\exists y\, \varphi(x, y)$ is logically equivalent to $\neg\forall y\, \neg\varphi(x, y)$, and we have shown that the $\forall$-formulas are closed under $\neg$ and $\forall$, up to $T$-equivalence.

(3)⇔(4): Suppose every formula is $T$-equivalent to a $\forall$-formula. Then for any formula $\varphi(x)$, the formula $\neg\varphi(x)$ is $T$-equivalent to a $\forall$-formula $\psi(x)$, so $\varphi(x)$ is $T$-equivalent to $\neg\varphi(x)$, which is logically equivalent to an $\exists$-formula. The converse is similar.

(4)⇒(1): Suppose $f\colon M \to N$ is an embedding, where $M, N \models T$. Let $\varphi(x)$ be a formula, and let $a \in M^x$. By (4) and its equivalent (3), $\varphi(x)$ is $T$-equivalent to both a universal formula $\psi_\forall(x)$ (which is reflected by $f$) and an existential formula $\psi_\exists(x)$ (which is preserved by $f$). So

$$\begin{aligned} M \models \varphi(a) &\implies M \models \psi_\exists(a) \\ &\implies N \models \psi_\exists(f(a)) \\ &\implies N \models \varphi(f(a)). \end{aligned}$$

And conversely,

$$\begin{aligned} N \models \varphi(f(a)) &\implies N \models \psi_\forall(f(a)) \\ &\implies M \models \psi_\forall(a) \\ &\implies M \models \varphi(a). \end{aligned}$$

□

This theorem captures the way in which model complete theories come very close to having quantifier elimination: every formula is equivalent to both a $\forall$-formula and an $\exists$-formula, but not necessarily a quantifier-free formula.

**Example 6.26.** Consider the theory of the real field, $T = \mathrm{Th}(\mathbb{R}; 0, 1, +, -, \cdot)$. The relation $x \leq y$ is $T$-definable by the formulas

$$\exists z\, (z \cdot z = y - x) \quad \text{or} \quad (x = y) \vee \forall z\, (z \cdot z \neq x - y),$$

but neither of these formulas is $T$-equivalent to a quantifier-free formula. Indeed, if they were equivalent to $\varphi(x, y)$, then the formula $\varphi(x, 0)$ would define $\{x \in \mathbb{R} \mid x \leq 0\}$. But $\varphi(x, 0)$ is a Boolean combination of polynomial equations in one variable $x$. Such a formula can never define an infinite and coinfinite set, since a polynomial in one variable has at most finitely many zeros.

In fact, the theory of the real field is model complete, which we may prove later in these notes, if time permits.

The condition that $M$ is existentially closed can be viewed as an abstraction of the condition that a field is algebraically closed. Indeed, the algebraically closed fields are exactly the existentially closed models of the theory of fields.

As an application, let's derive Hilbert's Nullstellensatz as a direct consequence of the model completeness of ACF.

**Theorem 6.27** (weak Nullstellensatz)**.** *Let $k$ be a field, let $K$ be its algebraic closure, and let $I$ be a proper ideal in the polynomial ring $k[x_1, \ldots, x_n]$. Then there is a tuple $(a_1, \ldots, a_n) \in K^n$ such that $p(a_1, \ldots, a_n) = 0$ for all $p \in I$.*

*Proof.* Let $\mathfrak{m}$ be a maximal ideal in $k[x_1, \ldots, x_n]$ extending $I$. Then the quotient ring $F = k[x_1, \ldots, x_n]/\mathfrak{m}$ is a field, and letting $b_i = x_i + \mathfrak{m}$ in $F$, we have $p(b_1, \ldots, b_n) = 0$ for all $p \in I$. Let $L$ be the algebraic closure of $F$. We have a natural embedding $k \to L$, and $L$ is algebraically closed, so we may assume that $k \subseteq K \subseteq L$, and by model completeness $K$ is existentially closed in $L$.

Now $I = (p_1, \ldots, p_m)$ is finitely generated, since $k[x_1, \ldots, x_n]$ is Noetherian (Hilbert's basis theorem). And $L \models \exists x_1 \ldots \exists x_n \bigwedge_{i=1}^m p_i(x_1, \ldots, x_n) = 0$. This is an $\exists$-formula with parameters from $k$ (the coefficients of the $p_i$), so also $K \models \exists x_1 \ldots \exists x_n \bigwedge_{i=1}^m p_i(x_1, \ldots, x_n) = 0$, as was to be shown. $\square$

We say theories $T$ and $T'$ are **companions** if every model of $T$ embeds in a model of $T'$ and vice versa. We say that $T^*$ is a **model companion** of $T$ if $T$ and $T^*$ are companions and $T^*$ is model complete.

**Proposition 6.28.** *If a theory $T$ has a model companion, then it is unique (up to equivalence).*

*Proof.* Suppose $T^*$ and $T'$ are both model companions of $T$. Then $T^*$ and $T'$ are companions (it is easy to see that the companion relation is transitive, in fact an equivalence relation on theories).

Let $M_0$ be a model of $T^*$. Then $M_0$ embeds in a model $N_0 \models T'$. And $N_0$ embeds in a model $M_1 \models T^*$. Continuing in this way, we build a chain:

$$\begin{tikzcd}
M_0 \arrow[rr, "\preceq"] \arrow[dr] & & M_1 \arrow[rr, "\preceq"] \arrow[dr] & & \cdots & M \arrow[d, dash, "\cong"] \\
& N_0 \arrow[ur] \arrow[rr, "\preceq"'] & & N_1 \arrow[ur] \arrow[r] & \cdots & N
\end{tikzcd}$$

Now letting $M = \varinjlim M_i$ and $N = \varinjlim N_i$, we have $M \cong N$. Since $T^*$ is model complete, $M$ is a directed colimit along a chain of elementary embeddings, so $\sigma_0 \colon M_0 \to M$ is an elementary embedding. Similarly, since $T'$ is model complete, $N$ is a directed colimit of models of $T'$ along a chain of elementary embeddings, so $N \models T'$. So $M_0 \preceq M \cong N \models T'$.

We have shown that every model of $T^*$ is a model of $T'$. The symmetric argument shows that $T^*$ and $T'$ are equivalent. $\square$

Tuesday 11/27 **Example 6.29.** The theory of fields and the theory of integral domains are companions. Their common model companion is ACF.

**Exercise 30.** Let $\mathcal{K}$ be the class of finite substructures of models of a theory $T$, and suppose $\mathcal{K}$ is a Fraïssé class. Then the generic theory $T_{\mathcal{K}}$ is the model companion of $T$. In particular, the model companion of $\mathrm{Th}(\mathbb{Z};<)$ is $\mathrm{Th}(\mathbb{Q};<)$.

**Exercise 31.** For any theory $T$, let $\mathcal{S}(T)$ be the class of all substructures of models of $T$, and let $T_\forall = \{\varphi \mid \varphi \text{ is a } \forall\text{-sentence, and } T \models \varphi\}$ be the set of all universal consequences of $T$. For any theories $T$ and $T'$, show that the following are equivalent:

(1) $T$ and $T'$ are companions.

(2) $\mathcal{S}(T) = \mathcal{S}(T')$.

(3) $T_\forall = T'_\forall$.

In the special case that $T$ is a $\forall\exists$-theory, we can characterize the model companion of $T$ in terms of the existentially closed models of $T$. The main point is the following proposition, which is an abstraction of the fact that every field embeds in an algebraically closed field.

**Proposition 6.30.** *Let $T$ be a $\forall\exists$-theory. Then every model of $T$ embeds in an existentially closed model of $T$.*

*Proof.* Let $M_0$ be a model of $T$. For some cardinal $\kappa$, let $(\varphi_\alpha)_{\alpha<\kappa}$ be an enumeration of all $\exists$-formulas with parameters from $M_0$.

We build a directed system $(M_0^\alpha)_{\alpha\leq\kappa}$ of models of $T$ indexed by the directed set $(\kappa+1,<)$, by transfinite induction. Let $M_0^0 = M_0$. For the successor step, given $M_0^\alpha$, if there is any $N \models T$ such that $M_0^\alpha \subseteq N$ and $N \models \varphi_\alpha$, then pick some such $N$ and let $M_0^{\alpha+1} = N$. Otherwise, let $M_0^{\alpha+1} = M_0^\alpha$.

When $\beta$ is a nonzero limit ordinal, we have $(M_0^\alpha)_{\alpha<\beta}$, such that $M_0^\alpha \subseteq M_0^{\alpha'}$ when $\alpha \leq \alpha'$, and $M_0^\alpha \models T$ for all $\alpha$. Let $M_0^\beta = \varinjlim M_0^\alpha$. Note that $M_0^\beta \models T$, since $T$ is $\forall\exists$ and hence preserved by directed colimits. Finally, let $M_1 = M_0^\kappa$.

Now we have $M_0 \subseteq M_1$, and for any existential formula $\varphi$ with parameters from $M_0$, if there exists $N \models T$ such that $M_1 \subseteq N$ and $N \models \varphi$, then already $M_1 \models \varphi$. Indeed, $\varphi = \varphi_\alpha$ for some $\alpha$, and $M_0^\alpha \subseteq M_0^{\alpha+1} \subseteq M_1 \subseteq N$, so $M_0^{\alpha+1} \models \varphi_\alpha$, and hence $M_1 \models \varphi_\alpha$, since $\varphi_\alpha$ is existential, and hence is preserved by embeddings.

But $M_1$ may not yet be existentially closed, since we've only handled existential formulas with parameters from $M_0$. So we repeat this process, building a chain $(M_i)_{i\in\omega}$, such that for any existential formula $\varphi$ with parameters from $M_i$, if there exists $N \models T$ such that $M_{i+1} \subseteq N$ and $N \models \varphi$, then already $M_{i+1} \models \varphi$. Let $M_\omega = \varinjlim M_i$ (again, $M_\omega \models T$ since $T$ is $\forall\exists$).

We claim that $M_\omega$ is existentially closed. Suppose $N \models T$ and $M_\omega \subseteq N$, and let $\varphi(a)$ be an existential formula with $a \in M_\omega^x$ such that $N \models \varphi(a)$. Then there is some $n$ such that $a \in M_n^x$. We have $M_n \subseteq M_\omega \subseteq N$, and by construction $M_{n+1} \models \varphi(a)$. So also $M_\omega \models \varphi(a)$, since $\varphi(x)$ is existential. $\square$

**Proposition 6.31.** *Suppose $T$ is a $\forall\exists$ theory, $N \models T$, and $M \subseteq N$. If $M$ is existentially closed in $N$, then $M \models T$.*

*Proof.* We show by induction on the construction of $\varphi(x)$ as a $\forall\exists$-formula that for all $a \in M^x$, if $N \models \varphi(a)$, then $M \models \varphi(a)$. The conclusion follows, when applied to the $\forall\exists$ sentences in $T$.

The base case, when $\varphi(x)$ is an $\exists$-formula, is just the fact that $M$ is existentially closed in $N$. And the inductive steps for $\wedge$ and $\vee$ are easy. So suppose the inductive hypothesis holds for $\varphi(x, y)$, and consider the formula $\forall y\, \varphi(x, y)$. If $a \in M^x$ and $N \models \forall y\, \varphi(a, y)$, then in particular for all $b \in M^y$, we have $N \models \varphi(a, b)$, so by induction $M \models \varphi(a, b)$, and $M \models \forall y\, \varphi(a, y)$. $\square$

**Theorem 6.32.** *Let $T$ be a $\forall\exists$-theory. Then $T$ has a model companion $T^*$ if and only if the class of existentially closed models of $T$ is an elementary class. In this case, $T^*$ is the theory of existentially closed models of $T$.*

*Proof.* Suppose first that the class of existentially closed models of $T$ is elementary, with theory $T^*$. We claim that $T^*$ is a model companion of $T$. By Proposition 6.30, every model of $T$ embeds in a model of $T^*$, and conversely every model of $T^*$ is itself a model of $T$. So $T^*$ and $T$ are companions. It remains to show that $T^*$ is model complete. But every model of $T^*$ is existentially closed in models of $T$, so in particular existentially closed in models of $T^*$, so we are done by Theorem 6.25.

Conversely, suppose $T$ has a model companion $T^*$. By Proposition 6.23, we may assume that $T^*$ is also a $\forall\exists$-theory. We claim that $M \models T^*$ if and only if $M$ is an existentially closed model of $T$.

If $M$ is an existentially closed model of $T$, then because $T$ and $T^*$ are companions, there are embeddings $M \subseteq N \subseteq M'$, where $N \models T^*$ and $M' \models T$. Then $M$ is existentially closed in $N$: if $\varphi(x)$ is existential, and $N \models \varphi(a)$, with $a \in M^x$, then $M' \models \varphi(a)$, so $M \models \varphi(a)$, since $M$ is existentially closed in $M'$. By Proposition 6.31, $M \models T^*$, since $T^*$ is $\forall\exists$.

In the other direction, if $M \models T^*$, we want to show that $M$ is an existentially closed model of $T$. So suppose $M \subseteq N$, where $N \models T$. Since $T$ and $T^*$ are companions, such an $N$ exists, and for any such $N$, there exists a model $M' \models T^*$ with $M \subseteq N \subseteq M'$. Since $T^*$ is model complete, $M$ is existentially closed in $M'$, so by the argument above, $M$ is existentially closed in $N$. By Proposition 6.31, $M$ is a model of $T$, since $T$ is $\forall\exists$, and since $N$ was an arbitrary model of $T$, $M$ is existentially closed. $\square$

**Exercise 32.** Let $T$ be the theory of torsion-free abelian groups. Show that $T$ has a model companion, and explicitly axiomatize it.

The following exercises show that not every theory has a model companion.

**Exercise 33.** Let $\mathcal{V} = (f, R)$, where $f$ is a unary function symbol and $R$ is a binary relation symbol. Consider the $\forall$-theory $T$ axiomatized by the following single sentence:

$$\forall x \forall y\, (R(x, y) \to R(x, f(y))).$$

(a) Show that for any existentially closed model $M \models T$ and any $a, b \in M$ with $b \notin \langle a \rangle$, we have $M \models \exists z\, (R(z, a) \wedge \neg R(z, b))$.

(b) Show that for any model $M \models T$ and any $a \in M$ such that $\langle a \rangle$ is infinite, there exists an elementary extension $M \preceq M'$, and some $b \in M'$ such that $b \notin \langle a \rangle$ and $M' \models \forall z\, (R(z, a) \to R(z, b))$.

(c) Show that the class of existentially closed models of $T$ is not elementary, so $T$ does not have a model companion.

**Exercise 34.** Show that the theory of groups does not have a model companion.

*Hint 1:* We say that elements $a$ and $b$ in a group $G$ are **conjugate** if there exists $c \in G$ such that $b = c^{-1}ac$. You may use as a black box the fact that for any group $G$ and any two elements $a, b \in G$ of the same order, there exists a group $H$ such that $G \subseteq H$ and $a$ and $b$ are conjugate in $H$.

*Hint 2:* Show that if a group $G$ contains elements of arbitrarily large finite order, then there is an elementary extension $G \preceq G'$ and elements $a, b \in G'$, each of infinite order, such that $a$ and $b$ are not conjugate in $G$.

# 7 Countable models

Thursday 11/29

Throughout this chapter, we assume that $\mathcal{V}$ is a countable vocabulary, so that $\mathcal{L}$ is countably infinite. Further, we assume that $T$ is a complete, consistent, interesting theory. We will study the category of countable models of $T$ and elementary embeddings.

We have already seen one kind of behavior that this category can have: in the $\aleph_0$-categorical case, it has a single object up to isomorphism. Our main examples of $\aleph_0$-categorical theories are Fraïssé limits. More generally, $T$ may have a **universal** countable model, into which every other countable model embeds elementarily. Or $T$ may have a **prime** model, which embeds elementarily into every other model.[20] The arguments will take inspiration from Fraïssé theory.

We will characterize the existence of universal and prime models, as well as $\aleph_0$-categoricity, in terms of properties of the type spaces $S_x(T)$, and we will characterize the universal and prime models themselves (as **saturated** and **atomic** models, respectively) in terms of the types they realize. We will show that the existence of a universal model implies the existence of a prime model. And as a curious application of all this theory, we will prove Vaught's "never two" theorem.

[20] But these are not terminal and initial objects, in general: the elementary embeddings will rarely be unique.

## 7.1 Saturated models

A countable model $M \models T$ is **saturated** if for all finite $A \subseteq_{\text{fin}} M$ and every single variable $x$, every type $p(x) \in S_x(A)$ is realized in $M$.

The condition that $A$ is finite is essential: the partial type $\{x \neq a \mid a \in M\}$ is consistent, hence extends to a complete type in $S_x(M)$, which is not realized in $M$.

**Example 7.1.** The algebraically closed field $F = \overline{\mathbb{Q}(x_0, x_1, x_2, \dots)}$ is a saturated model of $\mathrm{ACF}_0$. Indeed, suppose $A \subseteq F$ is finite. Let $K$ be the subfield of $F$ generated by $A$, and let $\overline{K}$ be its algebraic closure inside $F$. Let $p(x) \in S_x(A)$.

By quantifier elimination in $\mathrm{ACF}_0$, every formula is equivalent to a Boolean combination of polynomial equations $f(x) = 0$, where $f \in K[x]$. If $p(x)$ contains any positive polynomial equation $f(x) = 0$, then it is an **algebraic type** over $A$, which is realized in $F$ (by an element of $\overline{K}$). Otherwise, $p(x)$ must contain the formulas $f(x) \neq 0$ for all non-zero polynomials $f(x) \in K[x]$. This completely determines $p(x)$ has the unique **transcendental type** over $A$. It remains to show that this transcendental type is realized in $F$.

Note that $K$ is finitely generated over $\mathbb{Q}$ as a field, so $\overline{K}$ has finite transcendence degree over $F$, and $\overline{K}$ is a proper subfield of $F$. Since $\overline{K}$ is algebraically closed, any element of $F \setminus \overline{K}$ realizes the transcendental type over $A$.

Note the similarity between the definition of saturation and the definition of $\mathcal{K}$-homogeneity in the context of Fraïssé theory. The proofs in the next theorem should also look familiar.

**Theorem 7.2.** *Suppose $M \models T$ is a countable saturated model. Then:*

1. *$M$ is universal: If $N \models T$ is countable, then there is an elementary embedding $N \to M$.*
2. *If $M' \models T$ is another countable saturated model, $a \in M^x$, $a' \in (M')^x$ such that $\mathrm{tp}_M(a/\emptyset) = \mathrm{tp}_{M'}(a'/\emptyset)$, then there is an isomorphism $f\colon M \cong M'$ such that $f(a) = a'$.*
3. *$M$ is unique up to isomorphism.*

*Proof.* We first make the following observation. If $A \subseteq M$ is a set, $f\colon A \to M'$ is a partial elementary map, and $p \in S_x(A)$, then there is a type $f_*p \in S_x(f(A))$, defined by $\{\varphi(x, f(a)) \mid \varphi(x, a) \in p\}$.

The only thing to check is that $f_*p$ is consistent. If not, then by compactness, and since $p$ and hence $f_*p$ closed under conjunction, there is a formula $\varphi(x, f(a)) \in f_*p$ which is inconsistent. So $M' \models \neg\exists x\, \varphi(x, f(a))$. But since $f$ is partial elementary, also $M \models \neg\exists x\, \varphi(x, a)$. This means that $\varphi(x, a)$ is inconsistent, contradicting consistency of $p$.

For (1), we go forth, just as in the proof of Proposition 6.2. Enumerate $N$ as $(n_i)_{i\in\omega}$. Let $f_0$ be the empty function. We define a partial elementary map $f_i$ with domain $A_i = \{n_0, \dots, n_{i-1}\}$ for each $i \in \omega$ by induction. Given $f_i$, consider $p(x) = \mathrm{tp}(n_i/A_i)$. Since $f(A_i)$ is finite and $M$ is saturated, there is a realization $m_i$ of $(f_i)_*p \in S_x(f(A_i))$. Let $f_{i+1}$ extend $f_i$ by $f_{i+1}(n_i) = m_i$. Then $f_{i+1}$ is partial elementary, since for any formula $\varphi(x_0, \ldots, x_i)$, $N \models \varphi(n_0, \ldots, n_i)$ iff $\varphi(n_0, \ldots, n_{i-1}, x) \in p$ iff $\varphi(f(n_0), \ldots, f(n_{i-1}), x) \in f_*p$ iff $M \models \varphi(f(n_0), \ldots, f(n_{i-1}), f(n_i))$.

Finally, the union $f = \bigcup_{i \in \omega} f_i$ is an elementary embedding $N \to M$.

For (2), we only have to modify the proof of (1) to go back and forth, just as in the proof of Lemma 6.3.

(3) follows immediately from (2), since if $M' \models T$ is another countable saturated model, then $M \equiv M'$, so the empty tuples in $M$ and $M'$ realize the same types, and $M \cong M'$. $\square$

So saturated models are nice. Now we want to know when they exist. We say that $T$ is **small** if $|S_x(T)| \leq \aleph_0$ for all contexts $x$.

**Theorem 7.3.** *The following are equivalent:*

(1) *$T$ is small.*

(2) *$T$ has a saturated countable model.*

(3) *$T$ has a universal countable model.*

*Proof.* We have already proven (2) $\Rightarrow$ (3).

For (3) $\Rightarrow$ (1), suppose $T$ has a universal countable model $M$. Let $x$ be a context. For any type $p \in S_x(T)$, $p$ is realized by $a$ in some countable model $N \models T$, and there is an elementary embedding $f\colon N \to M$. Then $M \models p(f(a))$. So $M$ realizes every type in $S_x(T)$, but there are only countably many tuples in $M^x$, so there are only countably many types in $S_x(T)$.

For (1) $\Rightarrow$ (2), it remains to construct a saturated model. We begin with the observation that if $M \models T$ is any model and $A \subseteq_{\text{fin}} M$ is a finite subset, then $|S_x(A)| \leq \aleph_0$. Indeed, letting $y = (y_1, \ldots, y_n)$ be a tuple of variables enumerating $A = \{a_1, \ldots, a_n\}$, there is a map $S_x(A) \to S_{xy}(T)$ by replacing the parameters from $A$ by their corresponding variables in all formulas. Here is another way to describe this map: let $p(x) \in S_x(A)$, and let $a$ be a realization of $p(x)$ in some elementary extension $M'$ of $M$. Then $p(x) \mapsto \operatorname{tp}_{M'}(aa_1 \ldots a_n/\emptyset)$. This map is injective, so $|S_x(A)| \leq |S_{xy}(T)| \leq \aleph_0$, since $T$ is small.

Now let $M_0$ be any countable model of $T$. There are countably many finite subsets $A \subseteq_{\text{fin}} M_0$, and for each such $A$, the type space $S_x(A)$ is countable. So we can enumerate all of the types over finite subsets of $M_0$. Iteratively applying Proposition 5.3 and Löweheim–Skolem, we can find a countable elementary extension $M_0 \preceq M_1$ such that every type over a finite subset of $M_0$ is realized in $M_1$.

Repeating, we build an elementary chain $M_0 \preceq M_2 \preceq M_2 \preceq \ldots$. Letting $M_\omega = \varinjlim M_i$, we have $M_0 \preceq M_\omega$, so $M_\omega \models T$, and as a countable union of countable models, $M_\omega$ is countable. And $M_\omega$ is saturated, since if $A \subseteq_{\text{fin}} M_\omega$ and $p \in S_x(A)$, we have $A \subseteq M_k$ for some $k$, and $p$ is realized in $M_{k+1}$. $\square$

Note that it is not true in general that every universal countable model of $T$ is saturated. The fact is just that the *existence* of a universal countable model is equivalent to the *existence* of a saturated countable model.

Moving outside the realm of countable models for a moment: In general, for an infinite cardinal $\kappa$, we say that a model $M \models T$ is **$\kappa$-saturated** if for every $A \subseteq M$ such that $|A| < \kappa$ and every single variable $x$, every type $p(x) \in S_x(A)$ is realized in $M$. And we say that $M$ is **saturated** if it is $|M|$-saturated.

**Exercise 35.** Show that if $M \models T$ is $\kappa$-saturated, then $|M| \geq \kappa$.

**Example 7.4.** The complex field $\mathbb{C}$ is saturated. For any $A \subseteq \mathbb{C}$ with $|A| < |\mathbb{C}|$, let $K$ be the algebraic closure of the subfield generated by $A$. Then $|K| = \max(|A|, \aleph_0)$, so $|K| < |\mathbb{C}|$. Now as we argued above, there is one complete type in $S_x(A)$ for every element of $K$, and these are all realized in $\mathbb{C}$ by elements of $K$. And there is one additional type in $S_x(A)$, namely the transcendental type over $A$. This is realized by any element of $\mathbb{C} \setminus K$. Note that the same argument applies to any uncountable algebraically closed field.

**Example 7.5.** The real field $\mathbb{R}$ is not saturated (and not even $\aleph_0$-saturated). Indeed, for any $n > 0$, there is a formula without parameters expressing $x \leq \frac{1}{n}$. In the field language, this can be written as $\exists y\, (y^2 = 1 - n \cdot x)$, where $n$ is the term obtained by adding 1 to itself $n$ times. We can also express $0 < x$ by $(x \neq 0) \wedge \exists y\, (y^2 = x)$. Now consider the partial type

$$\{0 < x\} \cup \{x \leq \frac{1}{n} \mid n > 0\}.$$

This is consistent by compactness, so it extends to a complete type in $S_x(\emptyset)$, which is not realized in $\mathbb{R}$.

Tuesday 12/4

**Exercise 36.** (Some set theory involved) Show that for any theory $T$ and any infinite cardinal $\kappa$, $T$ has a $\kappa$-saturated model. *Hint:* You may assume $\kappa$ is a regular cardinal by replacing $\kappa$ by $\kappa^+$ if necessary. Then build an elementary chain of length $\kappa$.

**Exercise 37.** (Some more set theory involved) Show that for any theory $T$ and any uncountable regular cardinal $\kappa$ such that $\kappa^\lambda = \kappa$ for all cardinals $0 < \lambda < \kappa$, $T$ has a saturated model of cardinality $\kappa$.

In particular, if $\kappa$ is a strongly inaccessible cardinal, $T$ has a saturated model of cardinality $\kappa$. And if the continuum hypothesis holds, then $T$ has a saturated model of cardinality $2^{\aleph_0}$. But both of these set-theoretic assumptions (the existence of a strongly inaccessible cardinal, the continuum hypothesis) go beyond ZFC. It is consistent with ZFC that certain theories (the "unstable" ones, for example $\mathrm{Th}(\mathbb{R}; 0, 1, +, -, \cdot)$) have no saturated models at all.

## 7.2 Atomic models and omitting types

We have seen that to produce universal countable models, we need to realize as many types as possible. On the other end of the spectrum, to produce prime countable models, we need to realize as few types as possible.

Let $p(x) \in S_x(A)$ be a complete type. We say that $p$ is **isolated (over $A$)** if there is a formula $\varphi(x) \in L_x(A)$ such that $\varphi(x) \in p(x)$, and for every formula $\psi(x) \in p(x)$, $T \models \forall x\,(\varphi(x) \to \psi(x))$. In other words, $\varphi(x)$ completely determines the other formulas in $p(x)$, so $p(x)$ is the only complete type in $S_x(A)$ containing $\varphi(x)$. That is, the point $p \in S_x(A)$ is topologically isolated by the basic clopen set $[\varphi]$.

Note that if $A \subseteq M \models T$ and $p(x) \in S_x(A)$ is isolated over $A$ by the formula $\varphi(x, a)$, then since $p(x)$ is consistent, $M \models \exists x\,\varphi(x, a)$. Any witness to the existential quantifier realizes $p(x)$. The moral is: we can't avoid realizing isolated types.

A model $M \models T$ is **atomic** if it only realizes isolated types over $\emptyset$: for all contexts $x$ and all $a \in M^x$, $\mathrm{tp}(a) \in S_x(T)$ is isolated.

**Lemma 7.6.** *Let $a \in M^x$ and $b \in M^y$ be tuples from $M$. Then $\mathrm{tp}(ab) \in S_{xy}(T)$ is isolated if and only if $\mathrm{tp}(a) \in S_x(T)$ and $\mathrm{tp}(b/a) \in S_x(a)$ are both isolated.*

*Proof.* Suppose $\mathrm{tp}(ab)$ is isolated by $\varphi(x, y)$. Then $\mathrm{tp}(a)$ is isolated by $\exists y\,\varphi(x, y)$, and $\mathrm{tp}(b/a)$ is isolated by $\varphi(a, y)$.

Conversely, suppose $\mathrm{tp}(a)$ is isolated by $\psi(x)$ and $\mathrm{tp}(b/a)$ is isolated by $\chi(a, y)$. Then $\mathrm{tp}(ab)$ is isolated by $\psi(x) \wedge \chi(x, y)$. □

**Exercise 38.** Verify the assertions made in the proof of Lemma 7.6.

**Proposition 7.7.** *A model $M \models T$ is atomic if and only if for every finite $A \subseteq_{\mathrm{fin}} M$ and every element $b \in M$, $\mathrm{tp}(b/A)$ is isolated.*

*Proof.* Suppose $M \models T$ is atomic. For any $A \subseteq_{\mathrm{fin}} M$, and any $b \in M$, let $a$ be a finite tuple enumerating $A$. Then $\mathrm{tp}(ab)$ is isolated, so by Lemma 7.6, $\mathrm{tp}(b/A)$ is isolated.

Conversely, assume $\mathrm{tp}(b/A)$ is isolated for every finite $A \subseteq_{\mathrm{fin}} M$ and every $b \in M$. We show that for any tuple $a \in M^x$, $\mathrm{tp}(a) \in S_x(T)$ is isolated, by induction on the length of the tuple $a$.

In the base case, the tuple is empty. Since $T$ is complete, $S_{()}(T)$ (which is the space of completions of $T$) is a singleton, which is isolated.

Now suppose we are given a tuple $a = (a_1, \ldots, a_n)$. Let $a' = (a_1, \ldots, a_{n-1})$. By induction, $\mathrm{tp}(a')$ is isolated, and by assumption $\mathrm{tp}(a_n/a')$ is isolated, so by Lemma 7.6, $\mathrm{tp}(a)$ is isolated. □

**Example 7.8.** The algebraically closed field $F = \overline{\mathbb{Q}}$ is an atomic model of $\mathrm{ACF}_0$. We have seen that for any $A \subseteq_{\mathrm{fin}} F$, there is a unique transcendental type over $A$, and every other type is algebraic, i.e. contains a positive polynomial equation $f(x) = 0$ for some $f \in K[x]$, where $K$ is the subfield of $F$ generated by $A$.

Every such algebraic type is isolated. Indeed, if $p(x)$ is any complete type, then $I_p = \{f(x) \mid f(x) = 0 \text{ is in } p(x)\} \subseteq K[x]$ is a prime ideal in the polynomial ring. Since $K[x]$ is a PID, $I_p$ has a generator $g$, which is an irreducible polynomial, and the formula $g(x) = 0$ isolates the type $p(x)$. To see this, note that if $a, a'$ are any two realizations of the formula $g(x) = 0$ in any elementary extension of $F$, then $a$ and $a'$ are both roots of the irreducible polynomial $g(x)$, so they are conjugate by an automorphism (this is the fact that Galois groups act transitively on the roots of $g(x)$), so $\mathrm{tp}(a) = \mathrm{tp}(a') = p(x)$.

Every element of $F$ is algebraic over $\mathbb{Q}$, and hence over $K$ for any finitely generated subfield $K$, and it follows from Proposition 7.7 that $F$ is atomic. On the other hand, for any such $K$, the transcendental type over $K$ is not isolated and is not realized in $F$.

It turns out that atomic models are prime and satisfy the same uniqueness and homogeneity properties that saturated models do.

**Theorem 7.9.** *Suppose $M \models T$ is a countable atomic model. Then:*

1. *$M$ is prime: If $N \models T$, then there is an elementary embedding $M \to N$.*
2. *$M$ is unique up to isomorphism.*
3. *If $a, a' \in M^x$ such that $\mathrm{tp}(a) = \mathrm{tp}(a')$, then there is an automorphism $f\colon M \to M$ such that $f(a) = a'$.*

*Proof.* For (1), we go forth. Enumerate $M$ as $(m_i)_{i\in\omega}$. Let $f_0$ be the empty function. We define a partial elementary map $f_i$ with domain $A_i = \{m_0, \dots, m_{i-1}\}$ for each $i \in \omega$ by induction. Given $f_i$, consider $p(x) = \mathrm{tp}(m_i/A_i)$. Since $A$ is finite and $N$ is atomic, $p(x)$ is isolated by some formula $\varphi(x, a)$. Then $(f_i)_*p \in S_x(f(A_i))$ is also isolated by $\varphi(x, f_i(a))$, since $f_i$ is partial elementary. Indeed, for any formula $\psi(x, b) \in p(x)$, we have $M \models \forall x\,(\varphi(x, a) \to \psi(x, b))$, so $N \models \forall x\,(\varphi(x, f_i(a)) \to \psi(x, f_i(b)))$.

Thus there is a realization $n_i$ of $(f_i)_*p$ in $N$. Let $f_{i+1}$ extend $f_i$ by $f_{i+1}(m_i) = n_i$. Then $f_{i+1}$ is partial elementary, just as in the proof of Theorem 7.2. Finally, the union $f = \bigcup_{i\in\omega} f_i$ is an elementary embedding $M \to N$.

For (2) and (3), we only have to modify the proof of (1) to go back and forth, just as in the proof of Theorem 7.2. □

We are now faced with the challenge of constructing atomic models. This is a difficult point: the compactness theorem gives us a very flexible tool for realizing types in models of $T$. But how can we ensure that a given (non-isolated) type is **omitted** in a model of $T$? To solve this problem we need to return to a more "hands-on" approach to constructing models of $T$: the Henkin construction.

**Theorem 7.10** (Omitting types)**.** *Suppose $p \in S_x(T)$ is a non-isolated type. Then there is a countable model $M \models T$ such that $p$ is not realized in $M$.*

Before proving the theorem, let's note a consequence.

**Corollary 7.11.** *A model $M \models T$ is prime if and only if $M$ is countable and atomic.*

*Proof.* We have already shown that a countable atomic model of $T$ is prime.

Conversely, suppose $M$ is a prime model of $T$. By Löwenheim–Skolem, $T$ has a countable model $N$. Since $M$ embeds into $N$, $M$ is also countable.

Now let $a \in M^x$, and suppose for contradiction that $p(x) = \mathrm{tp}(a)$ is non-isolated. By omitting types, there is a model $N \models T$ such that $p(x)$ is not realized in $N$. But since $M$ is prime, there is an elementary embedding $f\colon M \to N$, and $f(a)$ realizes $p(x)$ in $N$, which is a contradiction. □

Thursday 12/6 *Proof of Theorem 7.10.* We will build a model of $T$ by constructing a complete consistent theory with Henkin witnesses, $T'$, extending $T$. Further, we will do this in a way that ensures that the canonical model of $T'$ does not realize the non-isolated type $p$.

Since $T$ is complete, for every sort $s$, $T$ decides whether $s$ is empty. That is, either $T \models \exists x\, \top$ (we say $s$ is **non-empty for** $T$) or $T \models \neg\exists x\, \top$ (we say $s$ is **empty for** $T$), where $x$ is a variable of type $s$. If $s$ is non-empty for $T$, let $C_s$ be a countably infinite set of new constant symbols of type $s$. Otherwise, let $C_s = \emptyset$. Let $C = (C_s)_{s\in\mathcal{S}}$, and let $\mathcal{V}' = \mathcal{V} \cup C$.

Rather than completing $T$ to a $\mathcal{V}'$-theory $T'$ by appealing to the ultrafilter lemma, we will build $T'$ "by hand"; this crucially uses the fact that the language $\mathcal{L}'$ is countable. In the construction of $T'$, we have three goals:

Goal 1: Ensure that $T'$ is complete (and consistent).

Goal 2: Ensure that $T'$ has Henkin witnesses.

Goal 3: Ensure that the canonical model of $T'$ does not realize $p$. We will do this by ensuring that for every tuple of constants $c \in C^x$, there is some formula $\varphi(x) \in p$ such that $\neg\varphi(c) \in T'$.

In order to achieve these goals, we make three lists.

List 1: Let $(\psi_i)_{i\in\omega}$ be an enumeration of all $\mathcal{V}'$-sentences. At stage $i$, we will add $\psi_i$ or $\neg\psi_i$ to $T'$.

List 2: Let $(\varphi_i(y))_{i\in\omega}$ be an enumeration of all $\mathcal{V}'$-formulas in a single free variable $y$. Further, ensure that each such formula $\varphi(y)$ occurs infinitely many times in the enumeration. At stage $i$, we will ensure that there is a Henkin witness for $\varphi_i(y)$.

List 3: Let $(c_i)_{i\in\omega}$ be an enumeration of $C^x$. At stage $i$, we will ensure that there is some formula $\varphi(x) \in p$ such that $\neg\varphi(c_i) \in T'$.

We now build a sequence of consistent $\mathcal{V}'$-theories $T_i$ by induction, such that $T_i$ contains $T$ together with only finitely many $\mathcal{V}'$-sentences. Let $T_0 = T$, and note that $T_0$ is a consistent $\mathcal{V}'$-theory, since any model of $T$ can be expanded to a $\mathcal{V}'$-structure by interpreting the new constants arbitrarily. For the inductive step, given the theory $T_i$, we build a theory $T_{i+1}$ by completing the tasks described above.

First, let $T_i' = T_i \cup \{\psi_i\}$ if this theory is consistent. Otherwise, let $T_i' = T_i \cup \{\neg\psi_i\}$, and note that this theory is consistent.

Second, if $T_i' \models \exists y\, \varphi_i(y)$, where $y$ has sort $s$, note that since $T \subseteq T_i'$ and $T$ is complete, $s$ is non-empty for $T$. Since $T_i'$ contains only finitely many $\mathcal{V}'$-sentences beyond $T$, it only mentions finitely many of the constant symbols in

$C_s$. Let $c_{\varphi_i} \in C_s$ be a constant symbol not mentioned in $T_i'$, and let $T_i'' = T_i' \cup \{\varphi_i(c_{\varphi_i})\}$.

We must check that $T_i''$ is consistent. So let $M \models T_i'$. Since $T_i' \models \exists y\, \varphi(y)$, there is some $a \in M$ such that $M \models \varphi(a)$. We can change the interpretation of the constant symbol $c$ to $a$ without changing the fact that $M \models T_i'$, since $c$ is not mentioned in $T_i'$. So the new structure $M'$ is a model of $T_i''$.

Third, we can list the finitely many new constant symbols mentioned in $T_i''$ as $(c_i, d)$, where $c_i$ is the tuple appearing in our enumeration of $C^x$, and $d$ consists of the remaining constant symbols not in $c_i$. Then we can write $T_i'' = T \cup \{\chi_j(c_i, d) \mid 1 \le j \le n\}$, where each $\chi_j$ is a $\mathcal{V}$-formula.

If the $\mathcal{V}$-formula $\chi(x) : \exists y \bigwedge_{j=1}^n \chi_j(x, y)$ is not in $p(x)$, let $T_{i+1} = T_i'' \cup \{\chi(c_i)\}$, and note that this is consistent, since any model of $T_i''$ satisfies the new sentence.

Otherwise, $\chi(x) \in p$. But since $p$ is non-isolated, $\chi(x)$ fails to isolate $p$. So there is some other complete type $q \in [\chi] \subseteq S_x(T)$ with $q \neq p$. Let $\varphi(x)$ be a formula such that $\varphi(x) \in p$ and $\neg\varphi(x) \in q$, and let $T_{i+1} = T_i'' \cup \{\varphi(c_i)\}$.

We must check in this case that $T_{i+1}$ is consistent. So let $M \models T$ be any model realizing $q$. Interpret the variables in $c_i$ as the tuple in $M^x$ realizing $q$. In particular, $M \models \varphi(c_i)$ and $M \models \chi(c_i)$. Now interpret the constant symbols in $d$ as the witnesses for the existential quantifiers in $\chi(x)$. So we have $M \models \bigwedge_{=1}^n \chi_j(c_i, d)$. Interpreting the rest of the constant symbols arbitrarily, $M \models T_{i+1}$.

We have completed our construction. We let $T' = \bigcup_{i\in\omega} T_i$, and we check that we have achieved our goals.

Goal 1: $T'$ is complete, since for every $\mathcal{V}'$-sentence $\psi$, $\psi$ appears as $\psi_i$ on List 1, and either $\psi_i$ or $\neg\psi_i$ was added to $T'$ in stage $i$. $T'$ is consistent (by compactness), since it is a union of consistent theories.

Goal 2: $T'$ has Henkin witnesses. Suppose $\varphi(y)$ is a $\mathcal{V}'$-formula in a single free variable $y$, such that $T' \models \exists y\, \varphi(y)$. Then by compactness there is a finite subset of $T'$ which entails $\exists y\, \varphi(y)$, so there is some $i$ such that $T_i \models \exists y\, \varphi(y)$. Now since $\varphi(y)$ appears infinitely many times on List 2, there is some $i^* \geq i$ such that $T_{i^*} \models \exists y\, \varphi(y)$ and $\varphi(y) = \varphi_{i^*}(y)$. By construction, $T_{i+1} \models \varphi(c_{\varphi_{i^*}})$, to $T' \models \varphi(c_{\varphi_{i^*}})$.

Goal 3: Let $M(T')$ be the canonical model of $T'$. We will show that $M$ does not realize $p$. The key observation is that every element of $M$ is named by a new constant symbol from $C$. Indeed, let $a \in M(T')_s$. Then $a$ is an equivalence class of terms of type $s$ in the empty context. Let $t$ be a representative of this class. Then letting $y$ be a variable of type $s$, $T' \models \exists y\, y = t$. So since $T'$ has Henkin witnesses, there is a constant symbol $c \in C_s$ such that $T' \models c = t$, and hence $c^{M(T')} = t^{M(T')} = a$, since every term evaluates in the canonical model to its equivalence class.

Now we observe that for any tuple $a \in M(T')^x$, we can pick some tuple of constants $c$ such that $c^{M(T')} = a$, and $c$ appears as $c_i$ on List

3. Then there is some formula $\varphi(x) \in p$ such that $T_{i+1} \models \neg\varphi(c)$, so $M(T') \models \neg\varphi(a)$, and hence $p$ is not realized in $M(T')$. □

We have shown how to omit a single non-isolated type. But to build an atomic model, we want to omit *every* non-isolated type. To do this, we need to improve the omitting types theorem.

Recall that a set $Y$ in a topological space $X$ is **nowhere dense** if for every open $U \subseteq X$, $Y$ is not dense in $U$. Equivalently, the closure of $Y$ has empty interior. A set $Y$ is **meager** if it is a countable union of nowhere dense sets.

In particular, a singleton set $\{p\}$ in a Hausdorff (or even $T_1$) space is nowhere dense if and only if $p$ is non-isolated. Indeed, if $p$ is isolated, i.e. $U = \{p\}$ is open, then $p$ is dense in $U$. Conversely, if $p$ is somewhere dense (not nowhere dense), then $\overline{\{p\}} = \{p\}$ has non-empty interior, i.e. $\{p\}$ is open, so $p$ is isolated. It follows that if $P \subseteq X$ is a countable set of non-isolated points, then $P$ is meager.

**Theorem 7.12** (Improved omitting types theorem)**.** *For each context $x$, let $P_x \subseteq S_x(T)$ be a meager set. Then there is a countable model $M \models T$ such that for all contexts $x$, no type in $P_x$ is realized in $M$.*

*Proof.* We follow the proof of the omitting types theorem. The main thing that changes is Goal 3: we now need to ensure that the canonical model of $T'$ does not realize any type in any set $P_x$. We will do this by ensuring that for every context $x$ and every tuple of constants $c \in C^x$, there is some formula $\varphi(x)$ such that $P_x \cap [\varphi] = \emptyset$ and $\varphi(c) \in T'$.

We also need to adjust our List 3. First, for each meager set $P_x$, we decompose $P_x = \bigcup_{n\in\omega} Q_{n,x}$, where $Q_{n,x}$ is nowhere dense. And instead of enumerating $C^x$, we enumerate triples $(x, c, Q_{n,x})$, where $x$ is a context, $c \in C^x$, and $Q_{n,x}$ appears in the decomposition of $P_x$.

At a particular stage, we need to deal with a triple $(x, c, Q_{n,x})$. We define the formula $\chi(x)$ in the same way, but this time we note that since $Q_{n,x}$ is nowhere dense, in particular it is not dense in $[\chi]$. So there is a basic open set $[\varphi] \subseteq [\chi]$ such that $[\varphi]$ is disjoint from $Q_{n,x}$. And we add $\varphi(c)$ to $T'$.

Finally, in the verification that $M(T')$ omits all of the types in the sets $P_x$, we note that if $x$ is a context, $c \in C^x$, and $p$ is a type in $P_x$, then $p \in Q_{n,x}$ for some $n$. Then there was some stage when we handled the triple $(x, c, Q_{n,x})$, so $\mathrm{tp}(c)$ contains a formula $\varphi(x)$ which is not contained in any of the types in $Q_{n,x}$. So $c$ does not realize $p$ in $M(T')$. □

**Corollary 7.13.** *Suppose $T$ has a countable saturated model. Then $T$ has a countable atomic model.*

*Proof.* By Theorem 7.3, $T$ is small. For every context $x$, let $P_x = \{p(x) \in S_x(T) \mid p(x) \text{ is non-isolated}\}$. Then since $|S_x(T)| \leq \aleph_0$, we have $|P_x| \leq \aleph_0$, and $P_x$ is a meager set. By the improved omitting types theorem, there is a model $M \models T$ omitting every non-isolated type. So $M$ is atomic. □

Corollary 7.13 gives a sufficient condition for the existence of a countable atomic model, in terms of the cardinality of the types spaces $S_x(T)$. The characterization of the existence of a countable atomic model has a more topological flavor. We say that **isolated types are dense** if the set of isolated types is dense in the type space $S_x(T)$ for every context $x$. That is, for every consistent formula $\varphi(x)$, there is an isolated type $p(x)$ such that $\varphi(x) \in p(x)$.

**Theorem 7.14.** *The following are equivalent, for a theory $T$:*

*(1) Isolated types are dense.*

*(2) $T$ has a countable atomic model.*

*(3) $T$ has a prime model.*

*Proof.* We have already proven (2)⇔(3).

For (1)⇒(2), let $P_x = \{p(x) \in S_x(T) \mid p(x) \text{ is non-isolated}\}$. Note that $P_x$ is closed, since its complement is the set of all isolated types, which is a union of open sets. And $P_x$ has empty interior, since if $U \subseteq P_x$ is a non-empty open set, then $U$ contains an isolated type, which is a contradiction. So $P_x$ is nowhere dense, hence meager. By the improved omitting types theorem, $T$ has a countable atomic model.

For (2)⇒(1), suppose $M \models T$ is prime. Let $\varphi(x)$ be a consistent formula. Then $T \models \exists x\, \varphi(x)$, so there is some $a \in M^x$ such that $M \models \varphi(a)$. Since $M$ is atomic, $p(x) = \mathrm{tp}(a)$ is an isolated type in $S_x(T)$ containing $\varphi(x)$. □

**Example 7.15.** Let $\mathcal{V} = ((R_n)_{n\in\omega})$, where each $R_n$ is a unary relation symbol. Consider the $\mathcal{V}$-structure $C$ with domain $2^\omega$, the set of all infinite binary sequences (equivalently the Cantor space), such that $R_n^C$ is the set of all sequences such that the $n^{\text{th}}$ term is 1. Let $T = \mathrm{Th}(C)$.

It is possible to show that $T$ has quantifier elimination, so a complete type $p(x)$ in one variable $x$ is determined by the set $\{n \mid R_n(x) \in p(x)\}$. It follows that $S_x(T)$ is homeomorphic to the Cantor space $2^\omega$. This space has no isolated types. So $T$ has no countable saturated model and no countable atomic model.

A countable model realizes only countably many types in $S_x(T)$, and we can use the improved omitting types theorem to build a countable model omitting the types in any meager subset of the Cantor space $S_x(T)$.

**Example 7.16.** Let $T = \mathrm{Th}(\mathbb{Q}; <, (q)_{q\in\mathbb{Q}})$, the theory of the rational order with a constant naming each element. By quantifier elimination, a type $p(x)$ in one variable $x$ is determined by the formulas of the form $x = q$, $x < q$, and $q < x$ in $p(x)$, for $q \in \mathbb{Q}$. In particular, for every $q \in \mathbb{Q}$, there is a type $p_q(x)$ isolated by the formula $x = q$. And for every downwards-closed set $L \subseteq \mathbb{Q}$, there is a non-isolated type $p_L(x)$ which contains $\{q < x \mid q \in L\} \cup \{x < q \mid q \notin L\}$.

Since there is one cut in $\mathbb{Q}$ for every real number, $T$ is not small, so it does not have a countable saturated model. But the isolated types are dense (a similar analysis applies to every variable context $x$), and $T$ has an atomic model, namely $\mathbb{Q}$.

## 7.3 $\aleph_0$-categorical theories

Thursday 12/13

As an additional appliation of the omitting types theorem, we will characterize $\aleph_0$-categorical theories. This is often called the Ryll-Nardzewski theorem, and additionally attributed to Engeler and Svenonius, all independently.

**Theorem 7.17.** *The following are equivalent, for a theory $T$:*

*(1) $T$ is $\aleph_0$-categorical.*

*(2) For all contexts $x$, every type in $S_x(T)$ is isolated.*

*(3) For all contexts $x$, $S_x(T)$ is finite.*

*(4) For all contexts $x$, $\mathcal{L}_x(T)$ is finite (i.e. there are only finitely many formulas in context $x$ up to $T$-equivalence).*

*Proof.* (1) $\Rightarrow$ (2): Let $M \models T$ be the unique countable model up to isomorphism. Then $M$ is prime, hence atomic. Let $p \in S_x(T)$. Then $p$ is realized in a countable model, which is isomorphic to $M$, so $p$ is realized in $M$, and $p$ is isolated.

(2) $\Rightarrow$ (1): Since every type relative to $T$ is isolated, every model of $T$ is atomic. But we proved that any two countable atomic models of a complete theory are isomorphic. So $T$ is $\aleph_0$-categorical.

(2) $\Rightarrow$ (3): Since every type in $S_x(T)$ is isolated, $\{\{p\} \mid p \in S_x(T)\}$ is an open cover of $S_x(T)$. By compactness, this open cover must be finite, so $S_x(T)$ is finite.

(3) $\Rightarrow$ (2): A finite $T_1$ space is discrete. More concretely: Let $p \in S_x(T)$. For every $q \in S_x(T)$ with $q \neq p$, pick some formula $\varphi_q(x) \in p$ such that $\neg\varphi_q(x) \in q$. Then $\bigwedge_{q \in S_x(T)} \varphi_q(x)$ isolates $p$.

(3) $\Leftrightarrow$ (4): $S_x(T)$ is the set of ultrafilters on the Boolean algebra $\mathcal{L}_x(T)$, and by Stone duality, $\mathcal{L}_x(T)$ is isomorphic to the clopen algebra of $S_x(T)$. So $S_x(T)$ is finite if and only if $\mathcal{L}_x(T)$ is. $\square$

We have now completed our classification of complete interesting theories in countable languages, according to the existence of prime and universal models, in terms of the topology and cardinality of the type spaces $S_x(T)$:

- $S_x(T)$ is finite for all $x$ $\Leftrightarrow$ every type in $S_x(T)$ is isolated: $T$ is $\aleph_0$-categorical, and the unique countable model is both prime and universal. Example: The theory of the random graph, see Example 6.10.
- $S_x(T)$ is countable for all $x$, and $S_x(T)$ is infinite for some $x$: $T$ has both a prime and a universal model, and these are different. Example: $\mathrm{ACF}_0$, see Examples 7.1 and 7.8.
- Isolated types are dense, and $S_x(T)$ is uncountable for some $x$: $T$ has a prime model but no universal model. Example: $\mathrm{Th}(\mathbb{Q}; <, (q)_{q \in \mathbb{Q}})$, see Example 7.16.

- Isolated types are not dense: $T$ has neither a prime model nor a universal model. Example: $\mathrm{Th}(2^\omega; (R_n)_{n\in\omega})$, see Example 7.15.

**Exercise 39.** Suppose $T$ is a single-sorted $\aleph_0$-categorical theory. Show that $T$ is *uniformly locally finite*: For every $n \in \omega$ there exists an $m \in \omega$ such that for any $M \models T$ and any $A \subseteq M$ with $|A| = n$, $|\langle A\rangle| \leq m$.

*Hint:* Suppose for contradiction that for some $n$, there is no bound on the size of substructures generated by sets of size $n$. Find an infinite set of formulas in $(n+1)$ variables which are pairwise $T$-inequivalent.

**Exercise 40.** Let $T_{\text{fields}}$ be the theory of fields, and let $T$ be a complete interesting theory with $T_{\text{fields}} \subseteq T$. Show that $T$ is not $\aleph_0$-categorical.

## 7.4 The number of countable models

Given a theory $T$ and a cardinal $\kappa$, we write $I(T,\kappa)$ for the number of models of $T$ of cardinality $\kappa$, up to isomorphism.

**Exercise 41.** (Assume $\mathcal{V}$ is countable.) For any theory $T$, $I(T,\kappa) \leq 2^\kappa$.

What are the possible values of $I(T,\aleph_0)$? Here is a summary of what we know so far:

- Any $\aleph_0$-categorical theory $T$ has $I(T,\aleph_0) = 1$.
- If $T = \mathrm{ACF}_p$ for $p$ prime or 0, then $I(T,\aleph_0) = \aleph_0$.
- If $T$ is a theory with $S_x(T) = 2^{\aleph_0}$ for some context $x$, such as the theories in Examples 7.16 and 7.15, then $I(T,\aleph_0) = 2^{\aleph_0}$. This is because every type is realized in some countable model, but any countable model can only realize countably many types.

Here is an example showing that $I(T,\aleph_0)$ can be finite but not 1.

**Example 7.18.** Let $T = \mathrm{Th}(\mathbb{Q}; <, (c_n)_{n\in\omega})$, where the constant symbol $c_n$ is interpreted as the natural number $n$. Then $T$ can be axiomatized by the theory of dense linear orders without endpoints, together with axioms $c_n < c_m$ when $n < m$. It is an exercise to show that these axiomatize a complete theory.

Now $T$ has exactly 3 countable models up to isomorphism:

1. Atomic: The sequence $(c_n)$ has no upper bound. Any such model is isomorphic to $(\mathbb{Q}; <, (c_n)_{n\in\omega})$, where $\lim_{n\to\infty} c_n = \infty$.
2. Saturated: The sequence $(c_n)$ has an upper bound, but no least upper bound. Any such model is isomorphic to $(\mathbb{Q}; <, (c_n)_{n\in\omega})$, where $\lim_{n\to\infty} c_n = \pi$.
3. Neither atomic nor saturated: The sequence $(c_n)$ has a least upper bound. Any such model is isomorphic to $(\mathbb{Q}; <, (c_n)_{n\in\omega})$, where $\lim_{n\to\infty} c_n = 1$.

**Exercise 42.** Let $n \geq 3$. Find an example of a theory $T$ such that $I(T,\aleph_0) = n$.

Curiously, the case $n = 2$ is impossible. It's worthwhile keeping Example 7.18 in mind while working through the proof.

**Theorem 7.19** (Vaught)**.** *There is no complete theory $T$ with exactly two countable models up to isomorphism.*

*Proof.* Let $T$ be a complete theory. We may assume $T$ is small, since otherwise it must have uncontably many countable models up to isomorphism. So $T$ has a countable saturated model $M_1$ and a countable atomic model $M_0$.

We may also assume $T$ is not $\aleph_0$-categorical. So there is some context $x$ and some non-isolated type $p(x) \in S_x(T)$. This type is realized in $M_1$ but not in $M_0$, so $M_0 \not\cong M_1$. Our goal is now to find a third model of $T$.

Let $a \in M_1^x$ be a realization of $p$, and consider the expanded language $\mathcal{V}(a)$ by new constants naming the elements of $a$. Let $M_1(a)$ be the expansion of $M_1$ to a $\mathcal{V}(a)$-structure, and let $T(a) = \mathrm{Th}_{\mathcal{V}(a)}(M_1(a))$.

Now $M_1(a)$ is still a countable saturated model of $T(a)$, because any $\mathcal{V}(a)$-type over a finite tuple $b$ from $M_1(a)$ is essentially the same as a $\mathcal{V}$-type over the finite tuple $ab$, and hence realied in $M_1$. So $T(a)$ is small, and hence it has a countable atomic model $M_{1/2}(a)$.

$T(a)$ is not $\aleph_0$-categorical by the Ryll-Nardzewski theorem, since any infinite family of $\mathcal{V}$-formulas which are pairwise not $T$-equivalent remain not $T(a)$-equivalent when viewed as $\mathcal{V}(a)$-formulas. So there is some context $y$ and some non-isolated type $q(y) \in S_y(T(a))$, which is omitted in $M_{1/2}(a)$.

Let $M_{1/2}$ be the reduct of $M_{1/2}(a)$ to $\mathcal{V}$ (forget about the new constant symbols). Then $M_{1/2}$ realizes $p(x)$, so it is not isomorphic to $M_0$. And $M_{1/2}$ omits $q(y)$, viewed as a type over the finitely many parameters $a$, so $M_{1/2} \not\cong M_1$. This is a contradiction. □

Returning to theories with infinitely many countable models, we have seen examples where $I(T, \aleph_0) = \aleph_0$ and $I(T, \aleph_0) = 2^{\aleph_0}$. Those interested in set theory will wonder about the possibility of cardinals between $\aleph_0$ and $2^{\aleph_0}$.

**Conjecture 7.20** (Vaught)**.** There is no theory $T$ such that

$$\aleph_0 < I(T, \aleph_0) < 2^{\aleph_0}.$$

Vaught's conjecture is one of the oldest open problems in model theory. Note that if we assume the continuum hypothesis, it is trivial, since there are no cardinals between $\aleph_0$ and $2^{\aleph_0} = \aleph_1$. The question is whether it is possible to prove Vaught's conjecture from the axioms of ZFC.

Using tools from descriptive set theory, Morley came close to settling the conjecture.

**Theorem 7.21** (Morley)**.** *There is no complete theory $T$ such that*

$$\aleph_1 < I(T, \aleph_0) < 2^{\aleph_0}.$$

In light of the theorems of Vaught and Morley, the possibilities for $I(T, \aleph_0)$ when $T$ is complete are: $1, 3, 4, 5, \ldots, \aleph_0, \aleph_1, 2^{\aleph_0}$. And Vaught's conjecture concerns the open case of $\aleph_1$.